Example closed non-orientable surface of genus g

cell structure: 1 0-cell

g 1-cells $a_1, a_2, \dots, a_g$ gluing map $a_1^2 a_2^2 \cdots a_g^2$.

1 2-cell

cellular chain complex $\mathbb{Z} \longrightarrow \mathbb{Z}^g \xrightarrow[0]{} \mathbb{Z}$

$1 \longmapsto (2,2,\dots,2)$

$H_2(M) = 0$

$H_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$

$H_0(M) = \mathbb{Z}$.

Euler characteristic

recall: surfaces (2-complexes) $\chi(S) = V - E + F$ $V = \#$vertices, $E = \#$edges, $F = \#$faces

 $4-6+4 = 2$.

fact: independent of triangulation of surface.

in general if X is a CW-complex, then $\chi(X) = \sum_n (-1)^n c_n$, $c_n = \#$cells in dimension n.

Thm $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$ (X finite dimensional)

recall: rank $= \#\mathbb{Z}$-summands / dimension of free part ($\dim \otimes \mathbb{Q}$)

exercise: $0 \to A \to B \to C \to 0$ exact, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof (purely algebraic)

let $0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots - C_1 \xrightarrow{d_1} C_0 \to 0$ be a chain complex

let cycles: $Z_n = \ker d_n$

boundaries: $B_n = \operatorname{im} d_{n+1}$

homology: $H_n = Z_n / B_n$

$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{d_{n-1}} B_{n-1} \to 0$

$0 \to B_n \to Z_n \to H_n \to 0$

short exact.

so $\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$

$\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n$

$\Rightarrow$ $\operatorname{rank} C_n = \operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}$

$\sum (-1)^n \operatorname{rank} C_n = \sum (-1)^n \operatorname{rank} H_n$, as required $\square$.

Homology with coefficients

$\mathbb{Z}$ - coefficients : consider chains $\sum n_i \sigma_i$, $n_i \in \mathbb{Z}$

G - coefficients : $\sum n_i \sigma_i$, $n_i \in G$ abelian group

still get a chain complex: $\rightarrow C_n(X;G) \xrightarrow{\partial_n} C_{n-1}(X;G) \rightarrow \cdots$

homology groups: $H_n(X;G)$ " homology with coefficients in G".

Example X = Moore space $M(\mathbb{Z}/m\mathbb{Z}, 1)$ = attach a 2-cell e^2 to S^1 by a map of degree m. Cellular homology chain: $0 \rightarrow \overset{2}{\mathbb{Z}} \xrightarrow{m} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \rightarrow 0$

$H_2(X)$ $H_1(X)$ $H_0(X)$

	dim 2	1	0
$H_*(X;\mathbb{Z})$	0	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}$
$H_k(X;\mathbb{Z}/m\mathbb{Z})$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$

consider quotient map $X \xrightarrow{f} X/S^1 \cong S^2$ induces $f_*: H_k(X) \rightarrow H_k(S^2)$.

note: $f_* = 0$ on $H_k(X;\mathbb{Z})$ for each k.

Q: is f homotopic to constant map?

A: no, consider long exact sequence of a pair with $\mathbb{Z}/m\mathbb{Z}$ coeffs. (X, S^1).

$$0 \rightarrow H_2(S^1;\mathbb{Z}/m\mathbb{Z}) \rightarrow H_2(X;\mathbb{Z}_m) \rightarrow H_2(\underset{\displaystyle S^2 =}{X/S^1}) \rightarrow H_1(S^1;\mathbb{Z}_m) \rightarrow \cdots$$

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow[f_*]{} \mathbb{Z}_m \qquad \mathbb{Z}_m$$

must be injective $\Rightarrow$ $f \not\simeq$ constant map.

§2.3 Formal view

Axioms for homology

A (reduced) homology theory assigns to each (non-empty) CW-complex X a sequence of abelian groups $\tilde{h}_n(X)$, and to each map $f: X \to Y$ a homomorphism $f_*: \tilde{h}_n(X) \to \tilde{h}_n(Y)$ such that $(fg)_* = f_* g_*$ and $1_* = 1$, s.t.

1) $f \simeq g : X \to Y$ then $f_* = g_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$.
2) there are boundary maps $\partial : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$ defined for each CW pair (X, A), with an exact sequence

$$\cdots \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

which are natural, i.e. given $f: (X, A) \to (Y, B)$ with $\bar{f}: X/A \to Y/B$ quotient map

s.t.

$$\begin{tikzcd} \tilde{h}_n(X/A) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"] & \tilde{h}_{n-1}(A) \arrow[d, "f_*"] \\ \tilde{h}_n(Y/B) \arrow[r, "\partial"] & \tilde{h}_{n-1}(B) \end{tikzcd}$$

commutes.

3) if $X = \bigvee_\alpha X_\alpha$, $i_\alpha : X_\alpha \hookrightarrow X$, then $\bigoplus_\alpha i_\alpha : \bigoplus_\alpha \tilde{h}_n(X_\alpha) \to \tilde{h}_n(X)$ isomorphism.

Examples $\tilde{H}_n(X)$ reduced singular homology, $\tilde{H}^{cw}_n(X)$ reduced cellular homology } (with coefficients)

Categories and functors

A category $\mathcal{C}$ consists of

1) a collection $Ob(\mathcal{C})$ of objects.
2) sets $Mor(X, Y)$ of morphisms for each pair $X, Y \in Ob(\mathcal{C})$ including an "identity" morphism $1_X \in Mor(X, X)$
3) composition of morphism function $\circ \; Mor(X, Y) \times Mor(Y, Z)$ for each $X, Y, Z \in Ob(\mathcal{C})$ s.t. $f \circ 1 = f$, $1 \circ f = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

Examples

- Ob = topological spaces, Mor(X,Y) cts maps from X to Y. (Top)
- groups homomorphisms (Group).
- sets functions
- chain complexes chain maps
- exact sequences.

Def^n A functor $F: \mathcal{C} \to \mathcal{D}$ (covariant)

$$\text{ob}: X \longrightarrow F(X)$$

$$\text{Mor}: f \in M(X,Y) \longmapsto F(f) \in M(F(X), F(Y)) \quad \text{s.t.} \quad F(1) = \mathbb{1}$$

$$F(f \circ g) = F(f) \cdot F(g)$$

Examples $\pi_1 : (\text{Top}, x_0) \to \text{Group}$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

H_n : Top $\longrightarrow$ Abelian groups

Pairs $\longrightarrow$ ~~chain~~ exact sequence

$(X,A) \longmapsto$ long exact sequence of a pair)

Def^n A functor is contravariant if $F(f \circ g) = F(g) \cdot F(f)$

Examples vector spaces $\longrightarrow$ vector space

$$V \longmapsto V^{\text{dual}}$$

top $\longrightarrow$ abelian group

cohomology. $X \longmapsto H^n(X)$

Def^n Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation T from F to G assigns a morphism $T_X : F(X) \to G(X)$, for each object X in $\mathcal{C}$, s.t. for each $f \in M(X,Y)$ s.t.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes.