Cellular homology

X CW-complex

useful facts a) $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$

$k=n$: free abelian with basis corresponding to n-cells of X.

b) $H_k(X^n) = 0$ for $k > n$

c) $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ for $k < n$

Proof a) (X^n, X^{n-1}) good pair, $X^n / X^{n-1} = \bigvee_\alpha S^n$, one for each n-cell α.

b) l.e.s. of pair (X^n, X^{n-1})

$$\cdots \to \underbrace{H_{k+1}(X^n, X^{n-1})}_{0 \text{ a)}} \to H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{0 \text{ a)}} \to \cdots$$

so $k > n$ then $H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots H_k(X^0) = 0$.

if $k < n$ then $H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots$ gives c) if X f.dim $\square$.

cellular chain complex: $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$

l.e.s. of pair (X^n, X^{n-1})

$$\begin{tikzcd}
 & & H_n(X^{n-1}) = 0 \arrow[d, "i_{n*}"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"'] & H_n(X^n) \arrow[r, "i_n"] \arrow[d, "j_{n*}"] & H_n(X^{n+1}) \arrow[r] & H_n(X^{n+1}, X^n) \\
 & & H_n(X^n, X^{n-1}) \arrow[d, "\partial_n"] \arrow[dr, "d_n"] & & \\
 & H_{n-1}(X^{n-2}) \arrow[r, "i_{n-1}"'] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"'] \arrow[d] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & \vdots & &
\end{tikzcd}$$

$H_n(X^{n+1}) = H_n(X)$, $H_n(X^{n+1}, X^n) = 0$, $H_{n-1}(X^{n-2}) = 0$, $H_n(X^{n-1}) = 0$.

define $d_{n+1} = j_n \partial_{n+1}$, $\boxed{d_n = j_{n-1} \partial_n}$, etc.

check: $d_n d_{n+1} = 0$

$j_{n-1} \underbrace{\partial_n j_n}_{0 \text{ (exact)}} \partial_{n+1}$

the homology groups of the chain complex $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \to \cdots$ are the <u>cellular homology groups</u> $H_n^{CW}(X)$

<u>Thm</u> $H_n^{CW}(X) \cong H_n(X)$ (if X is a CW-complex).

<u>Proof</u> $H_n(X) = H_n(X^n) / \operatorname{im} \partial_{n+1}$

diagram commutes, j_n injective, so $\operatorname{im} \partial_{n+1} \cong \operatorname{im}(j_n \partial_{n+1}) = \operatorname{im}(d_{n+1})$

also $H_n(X^n) \cong \operatorname{im} j_n = \ker d_n$

j_{n-1} injective, so $\ker \partial_n = \ker d_n$

therefore j_n gives an isomorphism $H_n(X^n)/\operatorname{im} \partial_{n+1} \cong \ker d_n / \operatorname{im} d_{n+1}$ □

<u>Applications</u>

1) $H_n(X) = 0$ if X is a CW complex with no n-cells.
2) X CW-complex with at most k n-cells, then $H_n(X)$ has at most k-generators.
3) if X has no cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis the cells.

<u>Examples</u> $\mathbb{C}P^n$ (one cell in each even dimension)

S^n

$S^n \times S^n$

Computing the boundary map

n=1: $d_1: H_1(X^1, X^0) \longrightarrow H_0(X^0)$ same as simplicial boundary map

$H_1^\Delta(X) \longrightarrow H_0^\Delta(X)$

$X^{(0)}$: . . $X^{(1)}$

special case: if X connected with 1-vertex then $d_1 = 0$ $| \mapsto 0$

n≥2: $d_n: H_n(X^n, X^{n-1}) \longrightarrow H_n(X^{n-1}, X^{n-2})$

(X^{n-1}, X^{n-2}) $\left(X^{n-1}/S_\beta, 1\right)$

n-cell e_α^n $B_\alpha^n \xrightarrow{\phi_\alpha} \bigvee_\beta^{R} S^{n-1}$

$\partial B^n = S^{n-1}$

$\phi_\alpha: \underset{S^{n-1}}{\partial B^n} \longrightarrow \bigvee_\beta S^{n-1} \longrightarrow S_\beta^{n-1}$

cellular boundary formula $\quad d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \xrightarrow[\phi_\alpha]{} X^{n-1} \longrightarrow S_\beta^{n-1}$

given by composition of the gluing map with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.

Example M_g closed orientable surface of genus g

1 0-cell

2g 1-cells $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$

1 2-cells.

gluing map: $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$

chain complex $\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow[0]{d_1} \mathbb{Z}$

each a_i, b_i appears twice with opposite sign $\Rightarrow d_2 = 0$.

so $H_k(M_g) = \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$.