<u>Application</u>

<u>Prop^n</u> $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely on S^n if n is even.

(G acts on $S^n \Leftrightarrow$ homomorphism $G \longrightarrow \mathrm{Homeo}(S^n)$).

<u>Proof</u> $\deg(f) = \pm 1$ if $f \in \mathrm{Homeo}(S^n)$, so we get $G \rightarrow \mathrm{Homeo}(S^n) \xrightarrow{d} \mathbb{Z}/2\mathbb{Z}$, homomorphisms. If G acts <u>freely</u> then no fixed points, so $d(f) = (-1)^{n+1} = -1$ if n even, so $\ker(d) = \{1\} \Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$ $\square$.

<u>How to compute degree</u>: pick generic $x \in S^n$ and count $f^{-1}(x)$ with signs:

$\deg(g) = 1$ [non-generic points: ]

Let $f: S^n \to S^n$ s.t. for some $y \in S^n$, $f^{-1}(y)$ consists of finitely many points $x_1, x_2, \dots, x_n$. Let $U_1, \dots, U_n$ be disjoint open neighbourhoods of $x_1, \dots, x_n$, s.t. $f(U_i) \subset V$, neighbourhood of y, s.t. $f(U_i \setminus x_i) \subset V \setminus y$ for each i. Write $f_*|_{x_i}$ for the local degree map $H_n(U_i, U_i \setminus x) \to H_n(V, V \setminus y)$ (if f is a local homeomorphism then $f_* = \pm 1$)

<u>Prop^n</u> $\deg(f) = \sum \deg(f|_{x_i})$

$$\begin{tikzcd}
& H_n(U_i, U_i \setminus x_i) \arrow[r, "f_*|_{x_i}"] \arrow[dl, "\text{excision } \cong"'] \arrow[d, "k_i(i_*)"] & H_n(V, V \setminus y) \arrow[d, "i_* \cong \text{ (excision)}"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
& H_n(S^n) \arrow[u, "j \text{ l.e.s of a pair}"'] \arrow[ul, "\cong", "\text{l.e.s of a pair}"'] \arrow[r, "f_*"', "\text{degree map}"'] & H_n(S^n) \arrow[u, "\cong \text{ l.e.s. of a pair}"'] \\
& \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

$H_n(S^n, S^n \setminus f^{-1}(y)) \cong \mathbb{Z}^{|f^{-1}(y)|}$, $H_n(S^n) \cong \mathbb{Z}$

claim: diagram commutes.

Prop^n $\deg f = \sum_i \deg f|_{x_i}$

Proof excision: $H_n(S^n, S^n \setminus f^{-1}(y)) = \bigoplus_i H_n(U_i, U_i \setminus x_i) \cong \mathbb{Z}^n$

~~diagram~~ diagram commutes:

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "\cong"', "\text{excision}"] \arrow[d, "k_i"] \\
\mathbb{Z} & \mathbb{Z}^n \arrow[l, "p_i"]
\end{tikzcd}
$$

k_i inclusion, p_i projection onto i-th factor.

$$
\begin{tikzcd}
\mathbb{Z} & \mathbb{Z}^n \arrow[l, "p_i"'] \\
 & \mathbb{Z} \arrow[u, "j"] \arrow[ul, "\cong"]
\end{tikzcd}
$$

$j: \mathbb{Z} \to \mathbb{Z}^n$, $1 \mapsto (1,1,\dots,1)$ so $p_i j(1) = 1$ for each i

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "f|_{x_i *}"] \arrow[d, "k_i"] & \mathbb{Z} \arrow[d, "i_*"] \\
\mathbb{Z}^n \arrow[r, "f_*"] & \mathbb{Z}
\end{tikzcd}
$$

$1 \mapsto \deg(f|_{x_i})$

$k_i(1) = (0,\dots,1,0,\dots,0)$ (ith coordinate)

$k_i(1) \mapsto \deg(f|_{x_i})$

$$
\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "f_*"] & \mathbb{Z} \\
\mathbb{Z} \arrow[u, "j"] \arrow[r, "f_*"] & \mathbb{Z} \arrow[u, "\cong \text{ l.e.s. pair}"']
\end{tikzcd}
$$

$(1,1,\dots,1) \mapsto \sum \deg(f|_{x_i})$

$1 \mapsto \deg(f_*)$

so $\deg(f_*) = \sum \deg(f|_{x_i})$.

Examples $z \mapsto z^d$ gives a map of degree d on S^1

suspend this to get a degree d map on S^k for any k.

Check this: (l.e.s. of a pair and naturality) $(S^k \times I, S^k \times \partial I = S^k \times \{0,1\})$.

$$0 \to \underset{0}{H_{k+1}(S^k \times I)} \to H_{k+1}(S^k \times I, S^k \times \partial I) \to H_k(S^k \times \partial I) \to H_k(S^k \times I) \to \underset{0}{H_k(S^k \times I, S^k \times \partial I)} \to 0$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "(f\times id)_*"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "f\times id_*"] & \mathbb{Z} \arrow[d, "(f\times id)_*"] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}
$$

$1 \mapsto (1,-1)$

$(a,b) \mapsto a+b$

$= (xd, xd)$.