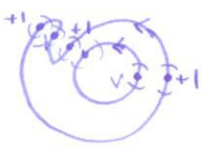
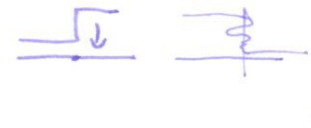


Application

Propⁿ $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely on S^n if n is even.
 (G acts on $S^n \iff$ homeomorphism $G \rightarrow \text{Homeo}(S^n)$).

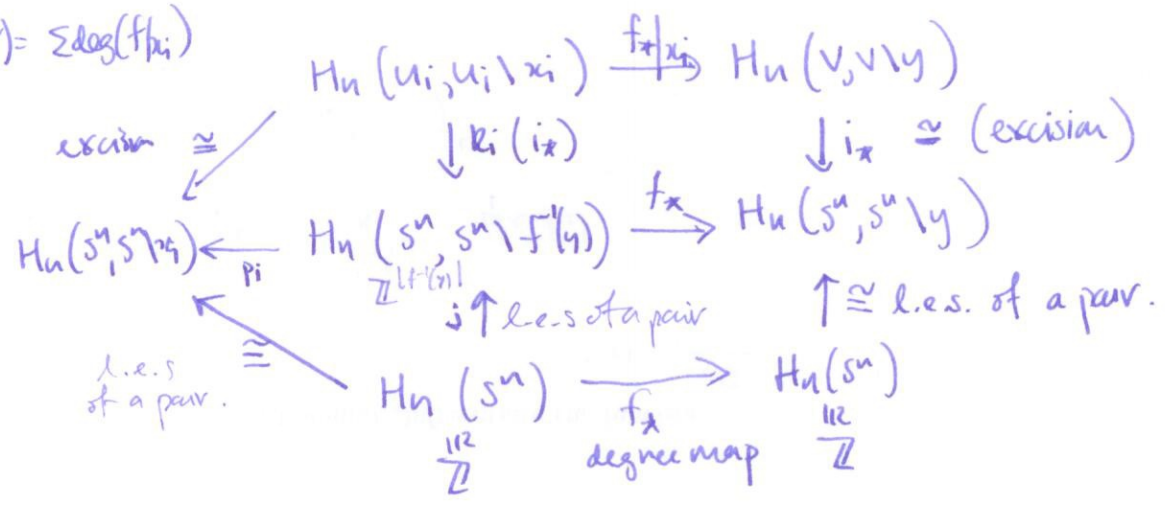
Proof $\deg(f) = \pm 1$ if $f \in \text{Homeo}(S^n)$, so we get $G \rightarrow \text{Homeo}(S^n) \xrightarrow{d} \mathbb{Z}/2\mathbb{Z}$,
 homeomorphisms. If G acts freely then no fixed points, so $d(f) = (-1)^{n+1} = -1$
 if n even, so $\ker(d) = \{1\} \Rightarrow G \cong \mathbb{Z}/2\mathbb{Z} \square$.

How to compute degree: pick generic $x \in S^n$ and count $f^{-1}(x)$ with

signs:  $\deg(g) = 1$ [non-generic points: 

Let $f: S^n \rightarrow S^n$ s.t. for some $y \in S^n$, $f^{-1}(y)$ consists of finitely many points
 $x_1, x_2, \dots, x_n$. Let $U_1, \dots, U_n$ be disjoint open neighborhoods of
 $x_1, \dots, x_n$, s.t. $f(U_i) \subset V$, neighborhood of y , $\&f(U_i \setminus x_i) \subset V \setminus y$
 for each i . Write $f_x|_{U_i}$ for the local degree map $H_n(U_i, U_i \setminus x_i) \rightarrow H_n(V, V \setminus y)$
 (if f is a local homeomorphism then $f_x = \pm 1$)

Propⁿ $\deg(f) = \sum \deg(f_{x_i})$



claim: diagram commutes.

Propⁿ $\deg f = \sum_i \deg f|_{x_i}$

Proof excision: $H_n(S^n, S^n \setminus f^{-1}(y)) = \bigoplus_i H_n(U_i, U_i \setminus x_i) \cong \mathbb{Z}^n$

~~excision~~ diagram commutes:

excision $\mathbb{Z} \xleftarrow{p_i} \mathbb{Z}^n$
 $\downarrow k_i$
 $\mathbb{Z} \xleftarrow{p_i} \mathbb{Z}^n$
 $\downarrow p_i$

k_i inclusion, p_i projection onto i -th factor.

$\mathbb{Z} \xleftarrow{p_i} \mathbb{Z}^n$
 $\uparrow j$
 $\mathbb{Z}$

$j: \mathbb{Z} \rightarrow \mathbb{Z}^n$
 $1 \mapsto (1, 1, \dots, 1)$ so $p_i j(1) = 1$ for each i .

$\mathbb{Z} \xrightarrow{f_x|_{x_i}} \mathbb{Z} \rightarrow \deg(f|_{x_i})$
 $\downarrow k_i$
 $\mathbb{Z}^n \xrightarrow{f_x} \mathbb{Z}$
 $k_i(1) \mapsto \deg(f|_{x_i})$

$k_i(1) = (0, \dots, 0, 1, 0, \dots, 0)$
 $\uparrow$
 i th coordinate.

$(1, 1, \dots, 1) \in \mathbb{Z}^n \xrightarrow{f_x} \mathbb{Z} \rightarrow \sum \deg(f|_{x_i})$
 $\uparrow j$
 $\mathbb{Z} \xrightarrow{f_x} \mathbb{Z}$
 $1 \mapsto \deg(f_x)$

$\uparrow \cong$ l.c.s. pair

so $\deg(f_x) = \sum \deg(f|_{x_i})$

Examples $\mathbb{Z} \xrightarrow{d} \mathbb{Z}^d$ gives a map of degree d on S^1

suspend this to get a degree d map on S^k for any k .

Check this: (l.c.s. of a pair and naturality). $(S^k \times I, S^k \times \partial I \rightarrow \{0, 1\})$.

$\partial \hookrightarrow H_{k+1}(S^k \times I) \rightarrow H_{k+1}(S^k \times I, S^k \times \partial I) \rightarrow H_k(S^k \times \partial I) \rightarrow H_k(S^k \times I) \rightarrow H_k(S^k \times I, S^k \times \partial I) \rightarrow 0$

$(f \times \text{id})_* \downarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1, -1)} \mathbb{Z} \xrightarrow{a+b \cdot (f \times \text{id})_*} \mathbb{Z}$
 $\downarrow f_* \downarrow = (x_d, x_d)$