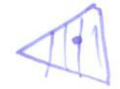
intuition $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) = \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n \end{cases}$

0 in $C_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$



s.t. x hits $\partial\Delta^n$.

Example find explicit cycles representing generators of $H_n(D^n, \partial D^n)$ and $H_n(S^n)$.

note: $(D^n, \partial D^n) = (\Delta^n, \partial\Delta^n)$

claim: $i_n: \Delta^n \to \Delta^n$ is a generator for $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

n=0: ✓

induction step: let $\Lambda \subset \Delta^n$ be the union of all but one $(n-1)$ dim faces.

consider $H_n(\Delta^n, \partial\Delta^n) \overset{\cong}{\underset{①}{\longrightarrow}} H_{n-1}(\partial\Delta^n, \Lambda) \overset{\cong}{\underset{②}{\longleftarrow}} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$

① long exact sequence of the triple $\Lambda \subset \partial\Delta^n \subset \Delta^n$

recall: $0 \to C_k(\partial\Delta^n, \Lambda) \to C_k(\Delta^n, \Lambda) \to C_k(\Delta^n, \partial\Delta^n) \to 0$.

gives: $\cdots \to H_k(\partial\Delta^n, \Lambda) \to H_k(\Delta^n, \Lambda) \to H_k(\Delta^n, \partial\Delta^n) \to H_{k-1}(\partial\Delta^n, \Lambda) \to \cdots$

Δ^n deformation retracts to Λ, so $H_k(\Delta^n, \Lambda) = 0$ for all k. $\triangleleft \leftarrow \Lambda$

so $H_k(\Delta^n, \partial\Delta^n) \overset{\cong}{\underset{\partial}{\longrightarrow}} H_{k-1}(\partial\Delta^{n-1}, \Lambda)$ for all k.

② consider $\Delta^{n-1} \hookrightarrow \partial\Delta$ as missing face.

$(\Delta^{n-1}, \partial\Delta^{n-1}) \hookrightarrow (\partial\Delta^n, \Lambda)$ [for good pairs $H_n(X,A) \underset{q}{\cong} H_n(X/A, A/A)$

$\downarrow q \qquad \downarrow q$

$\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$ homo S^n homo

therefore: a generator $i_n \in H_n(\Delta^n, \partial\Delta^n)$ is sent to to ∂i_n, a generator in $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$, so $\partial i_n = \pm i_{n-1} \in C_{n-1}(\partial\Delta^n, \Lambda)$

S^n: if $S^n = \Delta_1^n \cup \Delta_2^n$ by $[v_0, \dots, v_n] \leftrightarrow [w_0, \dots, w_n]$

then $\Delta_1^n - \Delta_2^n$ generates $H^n(S^n)$

exercise: $\tilde{H}_n(S^n) \xrightarrow[\text{pair}]{\cong} H_n(S^n, \Delta_2^n) \xleftarrow[\text{exc}]{\cong} H_n(\Delta_1^n, \partial \Delta_1^n)$ $\square$.

Equivalence of simplicial and singular homology

useful facts from homological algebra:

① naturality $f: (X,A) \to (Y,B)$ gives

$$\begin{array}{ccccccc} \cdots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y,B) & \to \cdots \end{array}$$

commutes.

② five lemma

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

rows exact, $\alpha, \beta, \delta, \varepsilon$ isos $\Rightarrow$ γ iso.

Thm (X,A) be a Δ-complex pair, the homomorphisms $H_n^\Delta(X,A) \to H_n(X,A)$ are isomorphisms, for all n.

Proof (X finite dimensional, $A = \emptyset$), let X^k = k-skeleton of X

(X^k, X^{k-1}) is a good pair.

$$\begin{array}{ccccccccc} \cdots \to H_{n+1}^\Delta(X^k, X^{k-1}) & \to & H_n^\Delta(X^{k-1}) & \to & H_n^\Delta(X^k) & \to & H_n^\Delta(X^k, X^{k-1}) & \to & H_{n-1}^\Delta(X^{k-1}) \to \cdots \\ \downarrow ① & & \downarrow ② & & \downarrow & & \downarrow ① & & \downarrow ② \\ \cdots \to H_{n+1}(X^k, X^{k-1}) & \to & H_n(X^{k-1}) & \to & H_n(X^k) & \to & H_n(X^k, X^{k-1}) & \to & H_{n-1}(X^{k-1}) \to \cdots \end{array}$$

① simplicial homology groups: $C_n^\Delta(X^k, X^{k-1}) = 0 \quad n \neq k$

$C_k^\Delta(X^k, X^{k-1})$ free abelian with basis Δ^k-simplices of X.

$= H_k^\Delta(X^k, X^{k-1})$.

singular homology groups: $H_n(X^k, X^{k-1}) \underset{\text{excision}}{\cong} H_n(X^k/X^{k-1}) \cong H_n\left(\bigvee_\alpha S^k\right)$

so $H_n(X^k, X^{k-1}) = 0$ for $k \neq n$

$H_k(X^k, X^{k-1})$ free abelian, with basis consisting of gluing maps $\phi_\alpha : \Delta^k \to X$ as $H_k(\Delta^k, \partial\Delta^k)$ generated by $i_n : \Delta^n \to \Delta^n$.

therefore $H_k^\Delta(X^k, X^{k-1}) \to H_k(X^k, X^{k-1})$ is an isomorphism.

② induction on k.

5 lemma. $\square$.

<u>Corollary</u> If X has a CW-complex structure with finitely many cells in each dimension, then $H_n(X)$ is finitely generated.

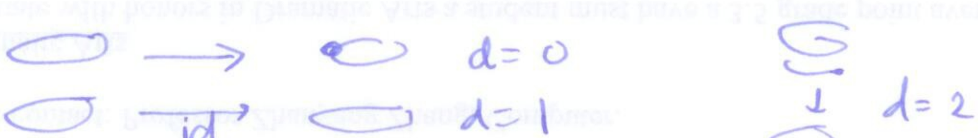
<u>Defn</u> rank(free part of $H_n(X)$) is called the <u>n-th Betti number</u> of X.

<u>§2.2 Applications</u>

<u>Defn</u> a map $f : S^n \to S^n$ induces $f_* : H_n(S^n) \to H_n(S^n)$, $\mathbb{Z} \to \mathbb{Z}$

$f_*(\alpha) = d\,\alpha$ for some $d \in \mathbb{Z}$. This integer is called the <u>degree</u>.

<u>Examples</u>



<u>useful properties</u>

a) $\deg(\mathbb{1}) = 1$ as $\mathbb{1}_* = \mathbb{1}$.

b) $\deg(f) = 0$ if f not surjective: can factor $S^n \to S^n \setminus \{x\} \to S^n$ (contractible. $H_n = 0$.)

c) $f \simeq g \Rightarrow \deg(f) = \deg(g)$ $\quad (f_* = g_*)$

d) $\deg(f \circ g) = \deg(f)\deg(g)$ $\quad \left((f g)_* = f_* g_* \right)$

in particular, if f is a homeomorphism or homotopy equivalence, then $\deg(f) = \pm 1$

e) $f : S^n \to S^n$ reflection in S^{n-1} then $\deg(f) = -1$

$S^n = \Delta_1^n \cup \Delta_2^n$ $\quad [v_0, \dots, v_n] \leftrightarrow [w_0, \dots, w_n]$ $\quad H_n(S^n)$ generated by $\Delta_1 - \Delta_2$,

reflection: $\Delta_1 - \Delta_2 \mapsto \Delta_2 - \Delta_1$.

f) antipodal map $-\mathbb{1} : S^n \to S^n$ has degree $(-1)^{n+1}$, as composition of $n+1$ reflections.

g) if $f : S^n \to S^n$ has no fixed points, then f is homotopic to $-\mathbb{1}$ by

$$f_t(x) = \frac{(1-t) f(x) - tx}{\| (1-t) f(x) - tx \|} \quad \text{so } \deg(f) = (-1)^{n+1}.$$

<u>Thm</u> S^n has a continuous field of non-zero tangent vectors iff n is odd.

<u>Proof</u> suppose $x \mapsto v(x)$ is a tangent vector field on S^n

(i.e. x and $v(x)$ are orthogonal).

If $v(x) \neq 0$ then we can normalize $\frac{v(x)}{\|v(x)\|}$.

Consider $v_t(x) = (\cos(t))x + (\sin(t))v(x)$ is a homotopy from the identity map $\mathbb{1}$ to the antipodal map $-\mathbb{1}$, i.e.

$\underset{\overset{\shortparallel}{1}}{\deg(\mathbb{1})} = \deg(-\mathbb{1}) = (-1)^{n+1} \Rightarrow n$ odd $\square$.

<u>Exercise</u>: construct $\underline{v}(x)$ on S^1, S^3.