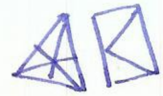
solution : 

define : $\rho(\sigma) = S^{m(\sigma)} - D_{m(\sigma)}\partial\sigma + \sum_{i=0}^{n} (-1)^i D_{m(\sigma_i)}\sigma_i$

[recall : $D_m = \sum_{i=0}^{m} TS^i$ and $\partial D_m + D_m\partial = \mathbb{1} - S^m$]

note : $\cancel{\rho : C_n(X) \to C_n^{\mathcal{U}}(X)}$. $\rho i = \mathbb{1}$ (don't need to subdivide) $m(\sigma) = 0$ for all σ.

define : $D : C_n(X) \to C_{n+1}(X)$ by $D\sigma = D_{m(\sigma)}\sigma$

claim : D is a chain homotopy between $\mathbb{1}$ and $i\rho$

proof : $\partial D\sigma + D\partial\sigma = \partial D_{m(\sigma)}\sigma + D\sum_{i=0}^{n}(-1)^i\sigma_i$

$$= \left(\mathbb{1} - S^{m(\sigma)} - D_{m(\sigma)}\partial\right)\sigma + \sum_{i=0}^{n}(-1)^i D_{m(\sigma_i)}\sigma_i$$

$$= \sigma - \underbrace{S^{m(\sigma)}\sigma - D_{m(\sigma)}\partial\sigma + \sum_{i=0}^{n}(-1)^i D_{m(\sigma_i)}\sigma_i}_{\rho(\sigma)} \qquad \square$$

Thm (excision) A, B, s.t. $X \subset \mathrm{int}(A) \cup \mathrm{int}(B)$

$(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$ for all n

Proof $\mathcal{U} = \{A, B\}$ write $C_n(A+B)$ for $C_n^{\mathcal{U}}(X)$ "chains in A + chains in B"

recall $\partial D + D\partial = \mathbb{1} - i\rho$ $\quad \rho i = \mathbb{1}$ (all maps take chains in $\underset{B}{A}$ to chains in $\underset{B}{A}$)

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(A+B)/C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, shift left, "i"] & C_n(A+B)/C_n(A) \arrow[r, "\partial"] \arrow[d, shift left, "i"] & C_{n-1}(A+B)/C_{n-1}(A) \arrow[r] \arrow[d, shift left, "i"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X)/C_{n+1}(A) \arrow[r] \arrow[u, shift left, "\rho"] & C_n(X)/C_n(A) \arrow[r] \arrow[u, shift left, "\rho"] & C_{n-1}(X)/C_{n-1}(A) \arrow[r] \arrow[u, shift left, "\rho"] & \cdots
\end{tikzcd}$$

$H_n(\underset{X}{A\cup B}, A)$

$H_n(X, A)$

all maps preserve $C_n(A)$ subchain of $C_n(A+B)$, so can quotient out note: $C_n(B) / C_n(A \cap B) \cong C_n(A+B) / C_n(A)$ (same basis of chains contained in $B \setminus A$).

so $H_n(B, A\cap B) \cong H_n(X, A)$, as required $\square$.

recall: (X,A) is a good pair if there is an open nbd $V \subset X$ s.t. V deformation retracts to A.

Thm: (X,A) good pair, then $q : (X,A) \to (X/A, A/A)$ induces isomorphisms
$$q_* : H_n(X,A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A) \text{ for all } n.$$

Proof ① long exact sequence of a triple (X, V, A)

$$\cdots \to H_n(V,A) \to H_n(X,A) \to H_n(X,V) \to \cdots$$

$\uparrow$ these are all zero, as $(V,A) \simeq (A,A)$ and $H_n(A,A) = 0$

② excision: $H_n(X, V) \cong H_n(X \setminus A, V \setminus A)$

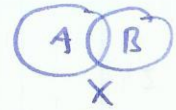
and $H_n(X/A, V/A) \cong H_n(X/A \setminus A/A, V/A \setminus A/A)$

so we get:

$$\begin{array}{ccccc}
H_n(X,A) & \underset{①}{\cong} & H_n(X,V) & \underset{②}{\cong} & H_n(X\setminus A, V\setminus A) \\
\downarrow q_* & & \downarrow q_* & & \downarrow q_* \text{(homeo!)} \\
H_n(X/A, A/A) & \underset{①}{\cong} & H_n(X/A, V/A) & \underset{②}{\cong} & H_n(X/A \setminus A/A, V/A \setminus A/A)
\end{array} \quad \square.$$

Corollary If X is a CW-complex, a union of subcomplexes A and B, then $(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X,A)$ for all n $\square$.

Corollary for a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce isomorphisms $\bigoplus_\alpha i_{\alpha *} : \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$ as long as the pairs (X_α, x_α) are good. $\square$



Mayer-Vietoris sequence (§2.2).

Thm: $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, get long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

recall: subdivision, set $\mathcal{U} = \{\mathrm{int}\, A, \mathrm{int}\, B\}$

$C_n^{\mathcal{U}}(X)$ = chains that are sums of chains in A and chains in B.

∂ preserves this, so $(C_n^{\mathcal{U}}(X), \partial)$ chain complex.

claim:

$$0 \to C_n(A \cap B) \xrightarrow{\phi = \text{"}i-j\text{"}} C_n(A) \oplus C_n(B) \xrightarrow{\psi = \text{"}i_A + i_B\text{"}} C_n^{\mathcal{U}}(X) \to 0$$
$$x \mapsto (x, -x)$$
$$(x, y) \mapsto x + y$$

is exact.

$$\begin{tikzcd} A\cap B \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & A \arrow[d, hook, "i_A"] \\ B \arrow[r, hook, "i_B"] & X \end{tikzcd}$$

proof (of claim).

- $\ker \phi = 0$ ✓ $(C_n(A \cap B) \subset C_n(A))$
- $\psi\phi(x) = \psi(x, -x) = x - x = 0$ so $\mathrm{im}\, \phi \subset \ker \psi$
- $\ker \psi \subset \mathrm{im}\, \phi$ spose $(x, y) \in C_n(A) \oplus C_n(B)$

  and $x - y = 0$ in $C_n^{\mathcal{U}}(X)$

  $\Rightarrow$ $x = y$ in $C_n^{\mathcal{U}}(X)$

  $\Rightarrow$ x, y are chains in $C_n^{\mathcal{U}}(A \cap B)$ $= C_n(A \cap B)$

  so lie in image of ϕ.
- ψ surjective by defn of $C_n^{\mathcal{U}}(X)$ □

Proof (of MV Thm) short exact sequence of chain complexes gives long exact sequence of homology groups. □.

<u>boundary map</u> : $H_n(X) \xrightarrow{\partial} H_{n-1}(A \cancel{\oplus} \cap B)$

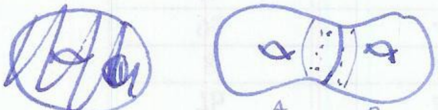
pick $\alpha \in H_n(X)$ represented by a cycle z, subdivide so $z = x+y$

where $\cancel{z}$ $x \in C_n(A)$, $y \in C_n(B)$ (not nec. cycles!)

but $\partial z = \partial x + \partial y = 0$ i.e. $\partial x = -\partial y$, but this implies

$\partial x = -\partial y \in C_{n-1}(A \cap B)$

so map $\alpha \mapsto \partial\alpha = [\partial x] = [-\partial y]$.

<u>Example</u> ① 

A B.

assume $H_k($$) = \mathbb{Z}$ $k=0$, $\mathbb{Z}^2$ $k=1$.

$A \cap B \simeq S^1$.

$$
\begin{array}{l}
\overset{0}{H_2(A\cap B)} \to \overset{0}{H_2(A)} \oplus \overset{0}{H_2(B)} \to \overset{\mathbb{Z}}{H_2(X)} \xrightarrow{\partial} \\
\overset{\mathbb{Z}}{H_1(A\cap B)} \xrightarrow{0} \overset{\mathbb{Z}^2}{H_1(A)} \oplus \overset{\mathbb{Z}^2}{H_1(B)} \to \overset{\mathbb{Z}^4}{H_1(X)} \xrightarrow{\partial} 0 . \\
\quad x \longmapsto (0, -0) \\
\overset{\mathbb{Z}}{H_0(A\cap B)} \to \overset{\mathbb{Z}}{H_0(A)} \oplus \overset{\mathbb{Z}}{H_0(B)} \to \overset{\mathbb{Z}}{H_0(X)} \\
\quad x \longmapsto (x, -x)
\end{array}
$$

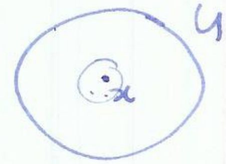
<u>Example</u> ② 

A B.

$$
\begin{array}{l}
\overset{0}{H_2(A\cap B)} \to \overset{\mathbb{Z}}{H_2(A)} \oplus \overset{0}{H_2(B)} \to \overset{\mathbb{Z}}{H_2(X)} \\
\overset{\mathbb{Z}}{H_1(A\cap B)} \xrightarrow{0} \overset{\mathbb{Z}^2}{H_1(A)} \oplus \overset{0}{H_1(B)} \to \overset{\mathbb{Z}^2}{H_1(X)} \to 0 . \\
\quad x \longmapsto (0, 0) . \\
\overset{\mathbb{Z}}{H_0(A\cap B)} \to H_0(A) \oplus \overset{\mathbb{Z}}{H_0(B)} \to \overset{\mathbb{Z}}{H_0(X)} . \\
\quad x \longmapsto (x, -x .
\end{array}
$$

Th$^{\underline{m}}$ (Brouwer) Invariance of domain. ($\mathbb{R}^n \not\cong_{\text{homeo}} \mathbb{R}^m$ unless $n=m$)

If non non-empty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n=m$.



<u>Proof</u> pick $x \in U$, and consider $H_k(U, U \setminus \{x\})$

$$\cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \quad \text{by excision}$$

long exact sequence of a pair:

$$\cdots \to H_n(\mathbb{R}^m \setminus \{x\}) \to \underset{0}{H_n(\mathbb{R}^m)} \to H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \to \cdots$$

$$\Rightarrow \quad H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^m \setminus \{x\})$$

$\uparrow$ deformation retracts to S^{m-1}

$$\Rightarrow \quad H_k(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & \text{for } k=m \\ 0 & \text{otherwise.} \end{cases}$$

a homeo $h: U \to V$ induces iso's on $H_k(U, U \setminus \{x\}) \to H_k(V, V \setminus \{h(x)\})$

$\Rightarrow m=n$ $\square$.

<u>**Def$^{\underline{n}}$**</u> The homology groups $H_n(X, X \setminus \{x\})$ are called the <u>local homology groups</u> of X at x.

<u>useful facts from homological algebra</u>

① <u>Naturality</u> the long exact sequence of a pair is <u>natural</u> as a map $f: (X,A) \to (Y,B)$ gives a commutative diagram

$$\begin{array}{ccccccc}
\cdots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to \cdots \\
& \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
\cdots \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y,B) & \to \cdots
\end{array}$$

② Five lemma:

$$\begin{array}{ccccccccccl}
A_1 & \to & B_1 & \to & C_1 & \to & D_1 & \to & E_1 & & \text{exact} \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon & & \\
A_2 & \to & B_2 & \to & C_2 & \to & D_2 & \to & E_2 & & \text{exact}
\end{array}$$

$\alpha, \beta, \delta, \epsilon$ isos $\Rightarrow \gamma$ iso.