define a subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$:

let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$

let $b \in \Delta^n$ be the barycenter, $\lambda(b)$ image of b in Y.

define : $\boxed{S(\lambda) = b_\lambda(S\partial\lambda)}$ where $b_\lambda : LC_{n-1}(Y) \to LC_n(Y)$

is the cone operator defined by the image of the barycenter $b(\lambda)$.

note: this is an inductive def$^{\underline{n}}$ (S on $LC_n(Y)$ defined in terms of S on $LC_{n-1}(Y)$).

base case $\underline{n = -1}$: $S[\phi] = [\phi]$ so $S = \mathbb{1}$ on $LC_{-1}(Y)$

$n = 0$: $S([w_0]) = b_{w_0} S[\phi] = b_{w_0}([\phi]) = [w_0]$

so $S = \mathbb{1}$ on $LC_0(Y)$

$n = 1$: $S([w_0, w_1]) = b_\lambda(S([w_1] - [w_0])) = b_\lambda([w_1]) - b_\lambda[w_0]$

$= [b, w_1] - [b, w_0]$.

and so on… $S \neq \mathbb{1}$ on $LC_1(Y)$

check : S is a chain map, i.e. $\partial S = S\partial$

$LC_n(Y) \xrightarrow{S} LC_n(Y)$

$n = 0, 1$ $S = \mathbb{1}$ ✓

general case : $\partial S(\lambda) = \partial(b_\lambda(S\partial\lambda))$ (def$^{\underline{n}}$ of S)

$= (\mathbb{1} - b_\lambda\partial)(S\partial\lambda)$ $(\partial b_\lambda = \mathbb{1} - b\partial_\lambda)$

$= S\partial\lambda - b_\lambda(\partial S\partial\lambda)$

$= S\partial(\lambda) - b_\lambda(S\underbrace{\partial\partial}_{=0}\lambda)$ (induction on n!)

$= S\partial(\lambda)$ as required.

want: chain homotopy T between S and $\mathbb{1}$

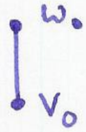
$$\begin{tikzcd}
\cdots \arrow[r] & LC_1(Y) \arrow[r] \arrow[dl, "T"'] \arrow[d, "S"', shift right] \arrow[d, "\mathbb{1}", shift left] & LC_0(Y) \arrow[r] \arrow[dl, "T"'] \arrow[d, "S"', shift right] \arrow[d, "\mathbb{1}", shift left] & LC_{-1}(Y) \arrow[r] \arrow[dl, "T"'] \arrow[d, "S"', shift right] \arrow[d, "\mathbb{1}", shift left] & 0 \\
\cdots \arrow[r] LC_2(Y) \arrow[r] & LC_1(Y) \arrow[r] & LC_0(Y) \arrow[r] & LC_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

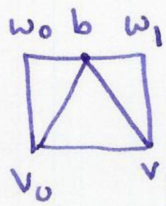
$T : LC_n(Y) \to LC_{n+1}(Y)$

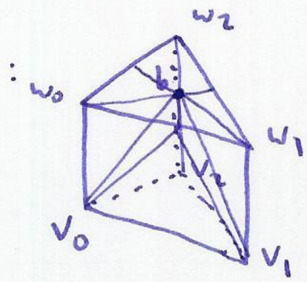
$n = -1 \; : \; T = 0$

$n \geq 0 \; : \; T\lambda = b_\lambda(\lambda - T\partial\lambda)$

why? subdivide $\Delta^n \times I$ by joining all simplices of $\Delta^n \times \{0\}$ ~~$\cup\, \partial\Delta^n \times I$~~ to the corresponding barycenter of in $\Delta^n \times \{1\}$, and each face in $T(\partial\Delta^n)$ to corresponding barycenter.

$n=0$:  $T = P$ prism operator.

$n=1$: 

$n=2$: 

<u>check</u> this is a chain homotopy : $\partial T + T\partial = \mathbb{1} - S$

<u>$n=-1$</u> : $T = 0$, $S = \mathbb{1}$ ✓

<u>$n \geq 0$</u> : $\partial T\lambda = \partial\big(b_\lambda(\lambda - T\partial\lambda)\big)$ (def^n of T).

$= (\mathbb{1} - b_\lambda\partial)(\lambda - T\partial\lambda)$ $\quad (\partial b_\lambda = \mathbb{1} - b_\lambda\partial)$

$= \lambda - T\partial\lambda - b_\lambda\partial\underbrace{(\mathbb{1} - T\partial)}_{=}\lambda$ $\quad$ (induction on n!).

$= \lambda - T\partial\lambda - b_\lambda\partial(S + \partial T)\lambda$

$$= \lambda - T\partial\lambda - b_\lambda \partial S\lambda - b_\lambda \underbrace{\partial\partial}_{0} T\lambda$$

$$= \lambda - T\partial\lambda - b_\lambda S\partial\lambda \qquad (\partial S = S\partial).$$

$$= \lambda - T\partial\lambda - S\lambda \qquad (\text{defn of } S)$$

<u>shown</u>: $\partial T = \mathbb{1} - T\partial - S$

$\partial T + T\partial = \mathbb{1} - S$ □.

③ Barycentric subdivision of general chains

key point: $\Delta^n \xrightarrow{\mathbb{1}} \Delta^n \xrightarrow{\sigma} X$

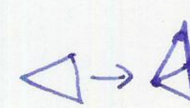
↑ identity is a linear map

aim: $C_n^{\mathcal{U}}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$

• barycentric subdivision for linear chains

cone operator: $b : [w_0, w_1, \dots, w_n] \longrightarrow [b, w_0, w_1, \dots, w_n]$

$b : LC_n(X) \to LC_{n+1}(X)$ s.t. $\partial b + b\partial = \mathbb{1}$

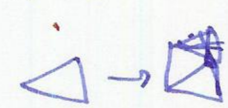


subdivision operator $S : LC_n(Y) \to LC_n(Y)$

$S(\lambda) = b_\lambda(S\partial\lambda)$ s.t. $\partial S = S\partial$

subdivided prism operator $T : LC_n(Y) \to LC_{n+1}(Y)$

$T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$ s.t. $\partial T + T\partial = \mathbb{1} - S$



define $S : C_n(X) \to C_n(X)$ i.e. $\Delta \xrightarrow{S} \Delta \xrightarrow{\sigma} X$.

$$\sigma \longmapsto \underbrace{\sigma_\#}_{\sigma|\text{subsimplex}} \underbrace{S\Delta^n}_{\text{sum of simplices in barycentric subdivision}}$$

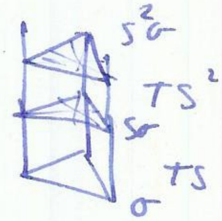
<u>claim</u>: S is a chain map $\partial S = S\partial$.

define $T : C_n(X) \to C_{n+1}(X)$ i.e. $\Delta^n \xrightarrow{T} \Delta^{n+1} \xrightarrow{\sigma} X$

$$\sigma \longmapsto \sigma_\# T\Delta^n$$

<u>claim</u>: T is a chain homotopy between S and $\mathbb{1}$, i.e. $\partial T + T\partial = S - \mathbb{1}$. $\square$

④ <u>iterated subdivisions</u>



<u>claim</u> $\mathbb{1}$ is chain homotopic to S^m by $D_m = \sum_{i=0}^{m} TS^i$

<u>proof</u>
$$\partial D_m + D_m\partial = \sum_i \partial T S^i + TS^i\partial \qquad (S \text{ chain map})$$
$$= \sum_i \partial TS^i + T\partial S^i$$
$$= \sum_i (\partial T + T\partial) S^i$$
$$= \sum_i (\mathbb{1} - S) S^i = \mathbb{1} - S + S - S^2 + \cdots - S^m \qquad \square.$$

for each $\sigma : \Delta^n \to X$ there is an $m(\sigma)$ s.t. $S^{m(\sigma)}\sigma \in C_n^{\mathcal{U}}(X)$, as $\sigma^{-1}(\operatorname{int}(U_i))$ is an open cover of $\Delta^n \subset \mathbb{R}^n$, so has a Lebesgue number $\epsilon > 0$. For each σ, let $m(\sigma)$ be the <u>smallest</u> such $m(\sigma)$

$$C_n^{\mathcal{U}}(X) \underset{p}{\overset{i}{\rightleftarrows}} C_n(X)$$ would like: $p(\sigma) = S^{m(\sigma)}\sigma$

doesn't work! (not a chain map)

problem: faces $\sigma_i = \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots v_n]}$ may have $m(\sigma_i) < m(\sigma)$.

(but $m(\sigma_i) \leq m(\sigma)$ for all σ_i)