<u>conclusion</u>: $A \subset X$

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \quad \text{exact}$$

if $[\alpha] \in H_n(X,A)$ represented by a relative cycle α, then

$\partial[\alpha] = [\partial\alpha]$ in $H_{n-1}(A)$.

<u>Useful facts</u>:

<u>Prop^n</u> If $f, g : (X,A) \to (Y,B)$ are homotopic by a pair-preserving homotopy ($F : X \times I \to Y$ s.t. $F(A,t) \subset B$) then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$ $\square$.

<u>Prop^n</u> Long exact sequence of a triple (X, A, B) $\quad B \subset A \subset X$

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots \quad \text{exact}$$

via: $\quad 0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$ exact. $\square$.

<u>Excision</u> $\quad Z \subset A \subset X$

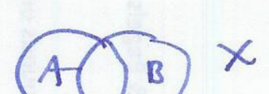
when does deleting $Z \subset A$ not change relative homology?

<u>Thm</u> $Z \subset A \subset X$ such that the closure of Z contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms

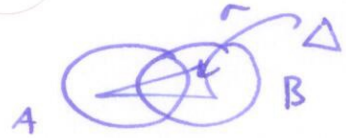
$$H_n(X \setminus Z, A \setminus Z) \xrightarrow[\cong]{i_*} H_n(X,A) \quad \text{for all } n.$$

<u>equivalently</u>: if $A, B \subset X$ and interiors of A, B cover X, then inclusions

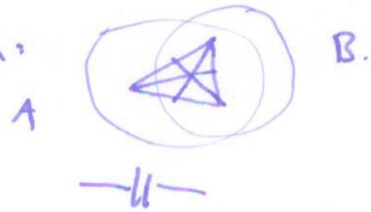
$(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms



$$H_n(B, A \cap B) \xrightarrow[i_*]{\cong} H_n(X,A) \quad \text{for all } n$$

basic problem: $X = A \cup B$  let $\sigma : \Delta^n \to X$ be a singular simplex

want to express σ as a sum of simplices in A, or in B.

solution: barycentric subdivision: 

Open covers

Let $\mathcal{U} = \{U_i\}$ be a collection of sets, whose interiors form an open cover of X

define: $C_n^{\mathcal{U}}(X) \subset C_n(X)$

$\uparrow$ subgroup consisting of chains $\sum n_i \sigma_i$ s.t. $\sigma_i(\Delta) \subset U_j$ for some U_j.

note:
$$\begin{array}{ccc} \partial : C_n(X) & \longrightarrow & C_{n-1}(X) \\ \cup & & \cup \\ C_n^{\mathcal{U}}(X) & \xrightarrow{\partial} & C_{n-1}^{\mathcal{U}}(X) \end{array}$$
so $C_n^{\mathcal{U}}(X)$ forms a chain complex

Prop$^{\underline{n}}$ The inclusion $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence, i.e. there is a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ s.t. $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity map.

Therefore ι induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.

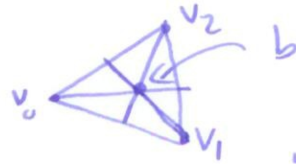
note:
$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d, shift right, "\rho"'] & C_{n-1}(X) \arrow[d, shift right, "\rho"'] \arrow[l, bend right, "F"'] \\ C_n^{\mathcal{U}}(X) \arrow[r, "\partial"] \arrow[u, shift right, "i"'] & C_{n-1}^{\mathcal{U}}(X) \arrow[u, shift right, "i"'] \arrow[l, bend left, "G"] \end{tikzcd}$$

want F s.t. $F\partial + \partial F = \iota\rho - \mathbb{1}$

G $\quad G\partial + \partial G = \rho i - 1$.

Proof

① Barycentric subdivision of simplices



simplex σ $[v_0, v_1, \dots, v_n]$ has barycentric coordinates $\sum_{i=0}^{n} t_i v_i$ $\quad \sum t_i = 1$, $t_i \geq 0$

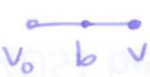
the barycenter of σ is $\quad b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$

we can decompose $[v_0, v_1, \dots, v_n]$ into simplices $[b, w_0, w_1, \dots, w_{n-1}]$

where each $[w_0, w_1, \dots, w_{n-1}]$ is a face of the barycentric subdivision of a face $[v_0, v_1, \dots, \hat{v}_i, \dots, v_n]$ of σ.

n=0 : $[v_0] \rightsquigarrow [v_0]$

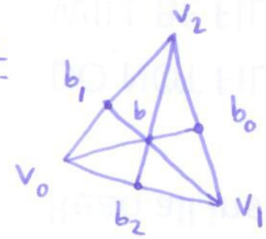
n=1 : $[v_0, v_1]$



$b = \frac{1}{2} v_0 + \frac{1}{2} v_1$ $\quad$ faces of $[v_0, v_1]$ are $[v_1], [v_0]$

so subsimplices are $[b, v_0]$ and $[b, v_1]$

n=2



get $[b, b_0, v_1]$ $\quad [b, b_1, v_0]$ $\quad [b, b_2, v_0]$

$[b, b_0, v_2]$ $\quad [b, b_1, v_2]$ $\quad [b, b_2, v_1]$.

claim : diameter of each subsimplex of Δ^n at most $\frac{n}{n+1}$ diameter of σ.

useful fact : diameter of a simplex = max distance between its vertices

proof (of ~~claim~~ useful fact) $\quad \underline{v} \in \Delta^n$ any vector $\quad \sum t_i v_i \in \Delta^n$

$$|v - \sum t_i v_i| = \left| \sum t_i (v - v_i) \right| \text{ as } \sum t_i = 1$$

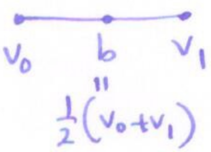
$$\leq \sum t_i |v - v_i| \quad \text{triangle inequality}$$

$$\leq \sum t_i \max |v - v_i|$$

$$\leq \max |v - v_i| \quad \sum t_i = 1. \quad \square.$$

<u>proof</u> (of claim) suffices to bound distances between vertices in the barycentric subdivision.

induction on n : base case $n=1$



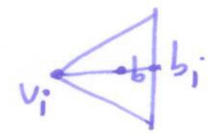
$d(b,v_i) \underset{(=)}{\leq} \frac{1}{2} d(v_0,v_1)$ ✓.

let $[w_0,\dots,w_n]$ be a simplex in the barycentric subdivision of $[v_0,\dots,v_n]$

~~if no $w_i = b$, then done by induction, as $[w_0,\dots,w_n]$ lives in a proper face of $[v_0,\dots,v_n]$~~. Let w_i, w_j be vertices realizing $\max |w_i - w_j|$.

if neither $w_i, w_j = b$, then done by induction, as they live in a proper face of $[v_0,\dots,v_n]$.

spse $w_j = b$. By convexity may take $w_i = v_i$ for some vertex v_i.



Let b_i be the barycenter of the opposite face to v_i, i.e. $[v_0,\dots,\hat{v}_i,\dots v_n]$

$$b = \sum \frac{1}{n+1} v_i \,, \quad b_i = \sum_{j \neq i} \frac{1}{n} b_j \qquad \text{so} \qquad b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$$

(so b lies in $[v_i, b_i]$). Therefore $|v_i - b| \leq \frac{n}{n+1} \underbrace{|v_i - b_i|}_{\text{both points in } \partial[v_0,\dots,v_n]}$

$$\leq \frac{n}{n+1} \operatorname{diam}([v_0,\dots,v_n]) \qquad \square$$

<u>note</u>: the diameter of the simplices in the r-th barycentric subdivision of Δ^n is at most $\left(\frac{n}{n+1}\right)^r \operatorname{diam}(\Delta^n) \longrightarrow 0$ as $r \to \infty$.

<u>important</u>: this bound does not depend on the <u>shape</u> of the simplices.

② Barycentric subdivision of linear chains

aim: construct a subdivision map $S : C_n(X) \to C_n(X)$ and show it is chain homotopic to the identity.

start with linear chains: $Y \subset \mathbb{R}^m$ convex, consider linear maps $\Delta^n \to Y$

generate a subgroup $\underset{\text{linear chains in } Y}{LC_n(Y)} \subset \underset{\text{chains in } Y}{C_n(Y)}$

note $\partial : LC_n(Y) \to LC_{n-1}(Y)$ so linear chains form a (sub) chain complex of $C_n(Y)$.

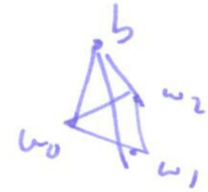
If $\lambda : \Delta^n \to Y$, $\lambda \in LC_n(Y)$ then λ determined by images of $[v_0, \dots, v_n]$ call them $w_i = \lambda(v_i)$, so image is $[w_0, \dots, w_n]$.

to deal with 0-simplices set $LC_{-1}(Y) = \mathbb{Z}$ generated by $[\phi]$ (empty simplex) $\partial [w_0] = [\phi]$ (recall ϵ).

a point $b \in Y$ determines a cone operator, i.e. a homomorphism

$b : LC_n(Y) \to LC_{n+1}(Y)$

$[w_0, \dots, w_n] \mapsto [b, w_0, w_1, \dots w_n]$



$$\partial b([w_0, \dots, w_n]) = \sum (-1)^i [b, \dots, \hat{w}_{i-1}, \dots, w_n]$$

$$= [w_0, \dots, w_n] - b(\partial [w_0, \dots, \hat{w}_i, \dots w_n])$$

extends linearly over linear chains:

if $\alpha = \sum n_i \sigma_i$, then $\partial b(\alpha) = \alpha - b \partial \alpha$

i.e. $\partial b(\alpha) + b \partial(\alpha) = \alpha$

i.e $\partial b + b \partial = \mathbb{1} - 0$

so b is a chain homotopy between $\mathbb{1}$ = identity and 0 map on $LC_n(Y)$