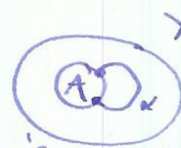
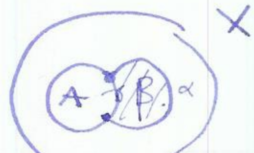
<u>note</u>: • element of $H_n(X,A)$ are represented by <u>relative cycles</u>

i.e. n-chains α in X s.t. $\partial\alpha \in C_{n-1}(A)$

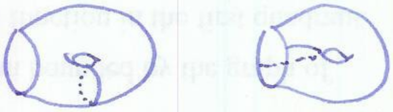


• a relative cycle α is trivial in $H_n(X,A)$ iff it is a <u>relative boundary</u>

i.e. $\alpha = \partial\beta + \gamma \qquad \beta \in C_{n+1}(X)$
$\qquad\qquad\qquad\qquad \gamma \in C_n(A)$



<u>Examples</u> typical choice of pairs are $(M,\partial M)$



$H_1(M) \qquad H_1(M,\partial M)$

<u>Claim</u> (long exact sequence of a pair)
there is a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \to \cdots$$

<u>Proof</u> ~~the maps~~ $A \overset{i}{\hookrightarrow} X$ consider:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[r, "\text{inclusion}"'] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[r, "\text{quotient}"'] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] \arrow[d] & C_{n-1}(X) \arrow[r, "j"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

← this is a short exact sequence.

← commutes!

in general, a short exact sequence of chain complexes is :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "\partial"] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where i and j are chain maps, i.e. diagram commutes

<u>FACT</u>: a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial"'] \\ & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

i_*, j_* homology maps induced by i, j.

<u>Q</u>: what is the map ∂? need a map $\partial : H_n(C) \to H_{n-1}(A)$

$$\begin{tikzcd} & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-2} \arrow[d, "i"] \\ b \in B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-2} \\ c \in C_n \arrow[r, "\partial"] & C_{n-1} & \end{tikzcd}$$

$a \mapsto \partial b$

pick a cycle $c \in C_n$ (i.e. $\partial c = 0$)
(defines a homology class $[c] \in H_n(C)$)

j onto $\Rightarrow \exists\, b \in B_n$ s.t. $j(b) = c$

consider $\partial b \in B_{n-1}$

note: $j(\partial b) = \partial j b = \partial c = 0$

so ~~[illegible]~~ $\partial b \in \ker(j) = \operatorname{im}(i)$, so there is $a \in A_{n-1}$ with $i(a) = \partial b$.

<u>claim</u>: $\partial a = a$ <u>proof</u>: $i \partial a = \partial i a = \partial \partial b = 0$
i.e. a is a <u>cycle</u> but i <u>injective</u> $\Rightarrow \partial a = 0$.

<u>Define</u>: $\partial : H_n(C) \longrightarrow H_{n-1}(A)$
$[c] \longmapsto [a]$ where a constructed as above.

<u>Claim</u>: this map is well defined.

<u>Proof</u>: check $[a]$ independent of choices

- a is uniquely determined by ∂b as i is injective.
- different choice b' for b would have

$j(b') = j(b)$, i.e. $j(b - b') = 0$

$b - b' \in \ker(j) = \operatorname{im}(i)$

so $b' = b + i(a')$, some $a' \in A_n$

but then $\partial b' = \partial b + \partial i(a')$

$$\begin{array}{ccccc} \partial b' & = & \partial b & + & \partial i(a') \\ \| & & \| & & \| \\ i(a'') & = & i(a) & + & i(\partial a') \end{array}$$

but then $[a''] = [a]$ as they differ by a boundary $\partial a'$.

• different choice of c, i.e. $c' = c + \partial c''$

$$\begin{array}{ccc} c' & = & c + \partial c'' \\ \| & & \| \\ j(b') & & j(b) \end{array}$$

so $j(b'-b) = \partial c''$, $\exists\, b''$ s.t. $j(b'') = c$

$\partial j(b'') = j\,\partial b''$ so we can chose $b' = b + \partial b''$

so $\partial b = \partial b'$, so choice of c doesn't matter □ (well defined)

<u>check</u>: $\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism.

spose $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ (via, b_1, b_2 resp, etc)

then $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$

and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$ so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$ □.

<u>Thm</u> $\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$ is exact.

<u>Proof</u> <u>check</u>: <u>proof</u>:

im $i_* \subset \ker j_*$ — $ji = 0 \Rightarrow j_* i_* = 0$ ✓

im $j_* \subset \ker \partial$ — $\partial j_* = 0$ as $\partial b = 0$ in definition of ∂ ✓

im $\partial \subset \ker i_*$ — $i_* \partial$ takes $[c]$ to $[\partial b] = 0$ so $i_* \partial = 0$ ✓

$\cancel{\ker j_* \supset \operatorname{im} i_*}$

$\ker j_* \subset \operatorname{im} i_*$

spose $[b] \in \ker j_*$ $\quad b \in B_n$, $\partial b = 0$

$\Downarrow$

$jb = \partial c$ for some $c \in C_{n+1}$

j surjective $\Rightarrow$ $c = j(b')$ for some $b' \in B_{n+1}$

$$j(b - \partial b') = j(b) - j(\partial b') = \underbrace{j(b)}_{\partial c} - \underbrace{\partial j(b')}_{\partial c} = 0$$

therefore $b - \partial b' = i(a)$ for some $a \in A_n$

claim: a is a cycle: i injective, so

consider $i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$

therefore $i_*[a] = [b - \partial b'] = [b]$

so i_* maps onto $\ker j_*$

$\ker \partial \subset \operatorname{im} j_*$:

spose $[c] \in \ker \partial$ then $a = \partial a'$ for some $a' \in A_n$

claim: $b - i(a')$ is a cycle:

$$\begin{aligned} \partial(b - i(a')) &= \partial b - \partial i(a') \\ &= \partial b - i \partial a' \\ &= \partial b - i a = 0 \end{aligned}$$

nb: $j(b - i(a')) = j(b) - ji(a') = j(b) = c$

so j_* maps $[b - i(a')]$ to $[c]$.

$\ker i_* \subset \operatorname{im} \partial$:

given $a \in A_{n-1}$ s.t. $i(a) = \partial b$ for some $b \in B_n$

then $j(b)$ is a cycle, as $\partial j(b) = j \partial b = ji(a) = 0$

and ∂ takes $[j(b)]$ to $[a]$. $\square$.