<u>Def^n</u> We say a chain map $P: C_n(X) \to C_{n+1}(Y)$ is a <u>chain homotopy</u> between two chain maps $f_\#, g_\#$, if $\partial P + P\partial = g_\# - f_\#$.

<u>Prop^n</u> chain homotopic chain maps induce the same homomorphism on homology.

Exact sequences and excision

a sequence of abelian groups and homomorphisms is <u>exact</u> if $\text{im}(\alpha_{n+1}) = \ker(\alpha_n)$ for all n. $\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$

(i.e. the chain complex with trivial homology).

<u>observations</u>

$0 \to A \xrightarrow{\alpha} B$ exact $\Rightarrow$ α injective

$A \xrightarrow{\alpha} B \to 0$ exact $\Rightarrow$ α surjective.

$0 \to A \xrightarrow{\alpha} B \to 0$ exact $\Rightarrow$ α isomorphism

$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact iff α injective, β surjective, $\ker \beta = \text{im}\, \alpha$

short exact sequence

so $C \cong B/\text{im}(\alpha) \cong B/A$

<u>excision</u>: $A \subset X$ can construct quotient space $X/A \longrightarrow X \setminus A \sqcup \{pt\}$.

"squash A to a point" topology on X/A: open sets in $X \setminus A$ (open sets in $X \setminus A$)

open sets in X containing A. (open sets in X containing $A \cup \{pt\}$)

if A is a subcomplex of X (cw-complex), just squash A to single 0-cell.

<u>example</u> $\partial D^2 \subset D^2$ $D^2/\partial D^2 = S^2$.

<u>Thm</u> Let A be a non-empty closed subspace of X, which is a deformation retract of an open set in X. Then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(X/A) \to \cdots$$

where i_* and j_* are induced by the maps $A \xhookrightarrow{i} X \xrightarrow{j} X/A$ and ∂ will be constructed during the proof.

<u>idea</u>: $x \in \tilde{H}_n(X/A)$ can be represented by a chain $\alpha \in C_n(X)$ with $\partial\alpha$ a cycle in $C_n(A)$, whose homology class is $\partial x \in \tilde{H}_{n-1}(A)$

<u>Defn</u> (Hatcher) (X,A) is a <u>good pair</u> if $A \subset X$, A non-empty, closed, and A is a deformation retract of an open set in X.

(i.e. excision works for good pairs)

in practice: $A \subset X$ normally A is CW-subcomplex of X.

<u>Application</u>: for S^n: $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u> $(X,A) = (D^n, \partial D^{n-1} \cong S^{n-1})$, so $X/A \cong S^n$

D^n contractible $\Rightarrow$ $\tilde{H}_n(D^n) = 0$ for all n. (i.e. every 3rd term zero in exact sequence). Therefore $\tilde{H}_k(S^n) \xrightarrow[\cong]{\partial} \tilde{H}_{k-1}(S^{n-1})$ isomorphisms for all $k > 0$. Now do induction on n starting with $S^0 = \{pt\} \cup \{pt\}$. $\square$.

Corollary Thm [Brouwer fixed point theorem]

∂D^n is not a retract of D^n. Therefore every map $f: D^n \to D^n$ has a fixed point.

recall: $A \subset X$ is a retract of X if there is a map $r: X \to A$ s.t. $r|_A = id_A = \mathbb{1}_A$

recall if there was a ~~retra~~ map $f: D^n \to D^n$ ~~could~~ with no fixed point then could construct retraction $r(x) = \frac{x - f(x)}{\|x - f(x)\|}$.

Proof suppose $r: D^n \to \partial D^n$ is a retraction. Then $\partial D^n \xhookrightarrow{i} D^n \xrightarrow{r} \partial D^n$ with $ir = \mathbb{1}_{\partial D^n}$, so consider induced map on homology: $H_{n-1}(\partial D^n) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(\partial D^n)$

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

$\Rightarrow (ir)_* = 0$ #. $\square$.

Relative homology

$A \subset X$ define $C_n(X, A) = C_n(X) / C_n(A)$
relative chain group

(i.e. chains in A are trivial in $C_n(X, A)$)

note: $\partial: C_n(X) \xrightarrow{\partial} C_{n-1}(X)$

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ \cup & & \cup \\ C_n(A) & \xrightarrow{\partial} & C_{n-1}(A) \end{array}$$

takes $C_n(A)$ into $C_{n-1}(A)$

so we get an induced boundary map $C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A)$,

so $\cdots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$ is a chain complex.

the homology groups of this chain complex are called the relative homology groups $H_n(X, A)$