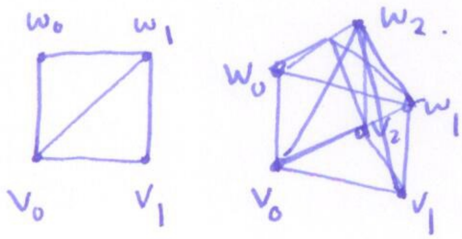
<u>Thm</u> If two maps $f, g : X \to Y$ are homotopic then they induce the same homomorphisms $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

<u>Proof</u> useful fact: we can divide $\Delta^n \times I$ into $(n+1)$-simplices

e.g.



let $\Delta^n \times \{0\} = [v_0, v_1, \dots, v_n]$

$\Delta^n \times \{1\} = [w_0, w_1, \dots, w_n]$

such that w_i projects to v_i under vertical projection.

consider the n-simplex $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$

this is the graph of the linear function $\phi_i : \Delta^n \to I$

$$(t_0, \dots, t_n) \longmapsto t_{i+1} + \dots + t_n$$

so $\quad 0 \leq \phi_n \leq \phi_{n-1} \leq \dots \leq \phi_1 \leq 1$

and the region between these graphs is the linear span of $[v_0, \dots, v_i, w_i, \dots, w_n]$

so is an $(n+1)$-simplex

therefore we have given $\Delta^n \times I$ an explicit ~~simplicial complex~~ Δ-structure structure

with $(n+1)$ $(n+1)$-simplices.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\# \, \downarrow \, g_\#"'] & C_n(X) \arrow[r] \arrow[dl, "P"'] \arrow[d, "f_\# \, \downarrow \, g_\#"] & {} \\ \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r] & {} \end{tikzcd}$$

let $F : X \times I \to Y$ be a homotopy from f to g

define a <u>prism operator</u> $\quad P : C_n(X) \to C_{n+1}(Y)$

by $\quad P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}) \Big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$

$\sigma : \Delta^n \to X$

$$\Delta^n \times I \xrightarrow{\sigma \times \mathbb{1}} X \times I \xrightarrow{F} Y$$

key fact: $\quad \partial P = g_\# - f_\# - P\partial$

"boundary of prism = top − bottom − sides".

proof

$$\partial P(\sigma) = \sum_{j\le i} (-1)^i (-1)^j F\circ(\sigma\times \mathbb{1})\Big|_{[v_0,\dots,\hat{v}_j,\dots v_i,w_i,\dots,w_n]}$$

$$+ \sum_{j\ge i} (-1)^i (-1)^{j+1} F\circ(\sigma\times \mathbb{1})\Big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots w_n]}$$

all the $i=j$ terms occur in pairs with opposite signs, except:

$$\underbrace{F\circ(\sigma\times\mathbb{1})\Big|_{[\hat{v}_0,w_0,\dots,w_n]}}_{g\circ\sigma \,=\, g_\#(\sigma)} \quad\text{and}\quad \underbrace{-F\circ(\sigma\times\mathbb{1})\Big|_{[v_0,\dots v_n,\hat{w}_n]}}_{-f\circ\sigma \,=\, f_\#(\sigma)}$$

the terms with $i\neq j$ are exactly $-P\partial$ as

$$\partial\sigma = \sum_j (-1)^j \sigma\Big|_{[v_0,\dots,\hat{v}_j,\dots,v_n]}$$

so

$$P\partial\sigma = \sum_{i<j} (-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})\Big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

$$+ \sum_{i>j} (-1)^{i-1}(-1)^j F\circ(\sigma\times\mathbb{1})\Big|_{[v_0,\dots,\hat{v}_j,\dots v_i,w_i,\dots w_n]}$$

Final step: spose $\alpha\in C_n(X)$ cycle, i.e. $\partial\alpha = 0$

then $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\underbrace{\partial(\alpha)}_{=0} = \partial P(\alpha)$ $\Rightarrow f_* = g_*$ on $H_n(X)$

i.e. $g_\#(\alpha) - f_\#(\alpha)$ is a boundary, so $[g_\#\alpha] = [f_\#\alpha]$ in homology

□