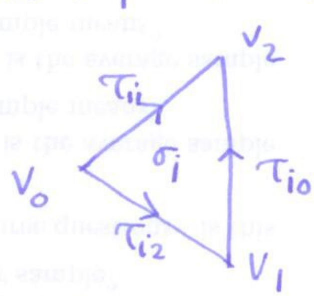
we have shown: $\pi_1(X, x) \twoheadrightarrow H_1(X)$

abelian, so commutator subgroup of $\pi_1(X,x) \subset \ker(h)$

<u>want</u>: $\ker(h) \subset$ commutator subgroup, i.e. we'll show if $[f] \in \ker(h)$ then $ab([f]) = 0$, where $ab : \pi_1(X,x) \longrightarrow ab(\pi_1(X,x))$.

spose $[f] \in \ker(h)$, then there is a 2-chain $\sum n_i \sigma_i$ s.t. $\partial(\sum n_i \sigma_i) = f$

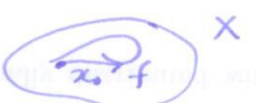
As before, can assume each $n_i = \pm 1$.



$\partial \sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$

so $f = \partial(\sum n_i \sigma_i) = \sum n_i \partial \sigma_i = \sum_{i,j} (-1)^j n_i \tau_{ij}$

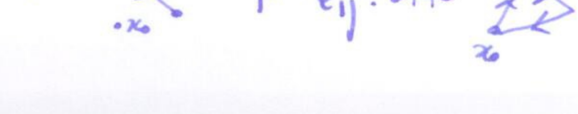
so we can pair off all of the τ_{ij}'s into pairs, where one has coeff $+1$, and the other has coeff -1, except for one left over corresponding to f



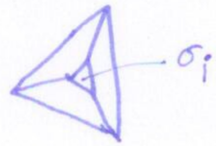
identify edges of the Δ^2_i's according to the pairing to produce a Δ-complex K, with a map $\sigma : K \to X$.

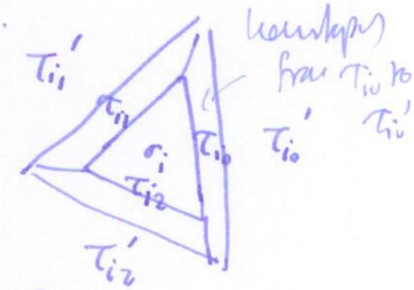
homotop σ, keeping σ fixed on the edge corresponding to f, such that all vertices map to x_0, as follows:

- for each vertex, choose a path to x_0. (determines homotopy on vertices).
- for each edge homotop τ_{ij}: $\longrightarrow$ to τ'_{ij}: $\gamma \sigma_i \bar{\gamma}$



– for each face Δ^2_j homotop $\sigma_i : \Delta \to$ 

to σ_i' : 



so now we have a new $\sigma' : K \to X$, with $\partial\sigma = f$, and all τ_{ij} map to loops based at x_0.

[note: $f \neq$ composition of τ_{ij} in $\pi_1 X$ in general]

but $[f]_{ab} = \sum_{i,j} (-1)^j n_i [\tau_{ij}]_{ab}$

↑ as can cancel ± 1 pairs in $(\pi_1(X, x_0))_{ab}$ to just get left with f.

$$= \sum n_i [\partial\sigma_i] = \sum n_i\big([\tau_{i0}] - [\tau_{i1}] + [\tau_{i2}]\big).$$

but the simplex $\sigma_i' : \Delta \to X$ gives a null homotopy of the compound loop corresponding to $\tau_{i0} - \tau_{i1} + \tau_{i2}$, so $[f]_{ab} = 0 \in \pi_1(X, x_0)_{ab}$.

$\square$.

<u>Recall</u> simplicial homology $\cdots \to C_n^{\Delta}(X) \to \cdots$ $H_n^{\Delta}(X)$

singular homology $\cdots \to C_n(X) \to \cdots$ $H_n(X)$

$H_0(X) \cong \mathbb{Z}^{\#\text{path components}}$

$H_1(X) = ab(\pi_1(X, x_0))$ (X path connected)

Homotopy invariance

Thm $X \simeq Y \Rightarrow H_n(X) = H_n(Y)$ for all n

$\uparrow$ homotopy equivalence.

Recall X, Y homotopy equivalent if there are maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t $gf \simeq \mathbb{1}_X$ and $fg \simeq \mathbb{1}_Y$.

Def^{n} a chain map $f : C_n(X) \to C_n(Y)$ s.t $\partial f = f \partial$.

Any cts map $f : X \to Y$ induces a chain map $f_\# : C_n(X) \to C_n(Y)$ by $f_\#(\sigma) = f\sigma : \Delta^n \to Y$ on singular simplices, then extend linearly to chains.

i.e. $f_\#\left(\sum_i n_i \sigma_i\right) = \sum_i n_i (f_\# \sigma_i)$.

import. **Prop^{n}** $f_\# \partial = \partial f_\#$

check: $f_\# \partial \sigma = f_\# \left(\sum (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$

$= \sum (-1)^i f\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \partial f_\# \sigma$. $\square$

so we get:

$$\begin{array}{ccccccccc}
\cdots \to & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \to \cdots \\
& \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & \\
\cdots \to & C_{n+1}(Y) & \xrightarrow[\partial]{} & C_n(Y) & \xrightarrow[\partial]{} & C_{n-1}(Y) & \to \cdots
\end{array}$$

commutes.

<u>note</u>: • $f_\#$ takes cycles to cycles

as if $\partial\alpha = 0$ then $\partial f_\# \alpha = f_\# \partial \alpha = f(0) = 0$

• $f_\#$ takes boundaries to boundaries

as if $\alpha = \partial\beta$, then $f_\# \alpha = f_\# \partial\beta = \partial f_\# \beta$.

we've shown:

<u>Prop^n</u> A chain map between chain complexes induces homomorphisms $f_*: H_n(X) \to H_n(Y)$ for all n.

<u>note</u>: • $(fg)_* = f_* g_*$

• $\mathbb{1}_* = \mathbb{1}$ check (exercise).

<u>Th^m</u> If two maps $f, g: X \to Y$ are homotopic, then they induce the same homomorphisms $f_* = g_*: H_n(X) \to H_n(Y)$ for all n.

<u>Corollary</u> $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ homotopy equivalence $\Rightarrow$ $H_n(X) \cong H_n(Y)$ for all n.

<u>Example</u> X contractible $\Rightarrow H_n(X) \cong H_n(\{pt\})$.