<u>Defn</u> reduced homology $\tilde{H}_n(X)$ (X non-empty)

as before, replace $\cdots \to C_0(X) \to 0$

with $\cdots \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$

$$\sum n_i \sigma_i \longmapsto \sum n_i$$

recall: $\epsilon \partial = 0$ so gives a chain complex.

in fact we get a map $H_0(X) \xrightarrow{\epsilon} \mathbb{Z}$ with kernel $\tilde{H}_0(X)$

so $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ $n=0$

$H_n(X) \cong \tilde{H}_n(X)$ $n \geq 1$

point of all this: $\tilde{H}_k(\cancel{X}\{pt\}) = 0$ for all k.

§ 2.A Homology and fundamental group

note: a loop $f : I \to (X, x_0)$ is also a singular 1-cycle. ($\partial f = 0$!).

as $\partial f = [x_0] - [x_0] = 0$.

<u>Thm</u> $h : \pi_1(X, x_0) \to H_1(X)$ is a homomorphism.

If X is path connected, then h is surjective and has kernel the commutator subgroup of $\pi_1(X, x_0)$, i.e. h gives an isomorphism $ab(\pi_1(X, x_0)) \cong H_1(X)$.

<u>Proof</u> notation: $f \simeq g$ homotopic rel endpoints.

$f \sim g$ homologous as chains.

note 1) f constant path, then $f \sim 0$.

check: • f is a cycle $\quad \partial f = [x_0] - [x_0] = 0$

• f is a boundary: let $\sigma : \Delta^2 \to x_0$ be the constant map

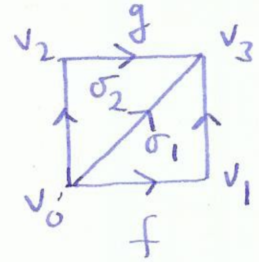
then
$$\partial \sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]}$$
$$= f - f + f = f$$

$H_1(X) = \ker(\partial_1) / \operatorname{im}(\partial_2)$ so $[f] = 0$ in $H_1(X)$.

2) homotopic $\Rightarrow$ homologous, i.e. $f \simeq g \Rightarrow f \sim g$

let $F : I \times I \to X$ be a homotopy from f to g

divide $I \times I$ into two triangles:



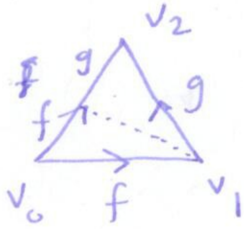
$\sigma_1 : [v_0, v_1, v_3]$

$\sigma_2 : [v_0, v_2, v_3]$

$$\partial(\sigma_1 - \sigma_2) = \sigma_1|_{[v_1,v_3]} - \cancel{\sigma_1|_{[v_0,v_3]}} + \sigma_1|_{[v_0,v_1]}$$
$$- \sigma_2|_{[v_2,v_3]} + \cancel{\sigma_2|_{[v_0,v_3]}} - \sigma_2|_{[v_0,v_2]}$$
$$= \begin{matrix} 0 + f \\ -g - 0 \end{matrix} = f - g$$

so $f - g$ is a boundary, so $f \sim g$.

3) $f \cdot g \sim f + g$: define $\sigma : \Delta^2 \to X$



by orthogonal projection onto $[v_0, v_2]$,
followed by $f \cdot g$ on $[v_0, v_2]$.

then $\partial X = g - f \cdot g + f$ so $f + g \sim f \cdot g$

4) $\bar{f} \sim -f$: just use $g = \bar{f}$ in above.

2), 3) give well defined homomorphism $h : \pi_1(X, x_0) \to H_1(X)$

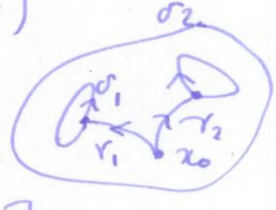
$$f \longmapsto [f]$$

<u>h surjective</u> (as long as X path connected)

let $\sum n_i \sigma_i$ be a 1-cycle representing an element of $H_1(X)$

- we may assume $n_i = \pm 1$ (just take repetitions of σ_i)
- we may assume $n_i = +1$ (replace σ_i with "reverse" of σ_i)

~~if path σ_i loops~~ suppose some σ_i is not a loop. as $\partial(\sum \sigma_i) = 0$
there must be some σ_j s.t. endpoint of σ_i is start of σ_j,
so replace $\sigma_i + \sigma_j$ with $\sigma_i \cdot \sigma_j$ (path composition)
continue until $\sum \sigma_i$ is a sum of loops



for each loop choose a path γ_i from x_0 to $\sigma_i|_{[v_0]}$

note: $\gamma_i \cdot \sigma_i \cdot \bar{\gamma}_i \sim \sigma_i$, and now all loops have basepoint x_0.

so $\gamma_1 \sigma_1 \bar{\gamma}_1 \, \gamma_2 \sigma_2 \bar{\gamma}_2 \cdots \gamma_n \cdot \sigma_n \bar{\gamma}_n \in \pi_1(X, x)$ which maps onto $\sum \sigma_i$,
as required $\square$.