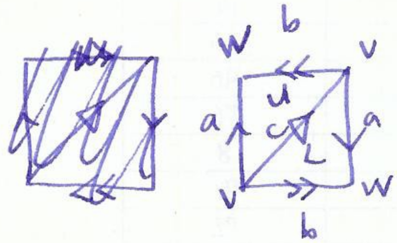
$$\left.\begin{aligned}\partial_2(u) &= a+b-c\\ \partial_2(L) &= a+b-c\end{aligned}\right\}$$ note $\{a, b, a+b-c\}$ is a basis for $\Delta_1(T)$

$$\Rightarrow \quad H_1^\Delta(T) \cong \mathbb{Z}^2$$
$$H_2^\Delta(T) \cong \mathbb{Z}$$
$$H_k^\Delta(T) = 0 \qquad k \geq 3.$$

(5) $\mathbb{RP}^2$



2 vertices
3 edges
2 faces

$$\Delta_3(X) \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$
$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0$$

$\partial_1(a) = w - v$
$\partial_1(b) = w - v$
$\partial_2(c) = v - v = 0$

$$H_0^\Delta(X) = \langle w, v\rangle / \langle w - v\rangle \cong \mathbb{Z}$$

$\partial_2(u) = a - b - c$
$\partial_2(L) = a - b + c$

$$H_1^\Delta(X) = \ker(\partial_1)/\operatorname{im}(\partial_2)$$

$\ker(\partial_1)$ has basis $\{a-b, c\}$ or $\{a-b+c, c\}$. $\cong \mathbb{Z}^2$.

$\operatorname{im}(\partial_2) \cong \mathbb{Z}^2$ has basis $\{a-b+c, 2c\}$

so $\quad H_1^\Delta(X) \cong \mathbb{Z}/2\mathbb{Z}$

$$H_2^\Delta(X) = \ker(\partial_2)/\operatorname{im}(\partial_3) = 0.$$

$$H_n^\Delta(X) = \begin{cases} 0 & n \geq 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

6. $S^n = \underset{U}{\Delta^n} \cup \underset{L}{\Delta^n}$ gluing map = id

$$\begin{array}{ccccc} \Delta_{n+1}(S^n) & \to & \Delta_n(S^n) & \to & \cdots \\ 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\partial_n]{} & \end{array}$$

$\ker(\partial_n)$ generated by $U-L$

$\Rightarrow H_n^\Delta(S^n) \cong \mathbb{Z}$

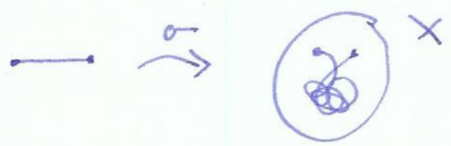
<u>Q</u>: · is $H_n^\Delta(X)$ independent of choice of Δ-structure?

· is $H_n^\Delta(X)$ a homotopy invariant?

<u>singular homology</u>

<u>Def$^{\underline{n}}$</u> A <u>singular n-simplex</u> is a map $\sigma : \Delta^n \to X$

e.g. $\Delta^1 =$



Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$ are called (singular) <u>n-chains</u>

i.e. formal sums $\sum_i n_i \sigma_i$ where $\sigma_i : \Delta^n \to X$

Define the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]}$$

where $\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ is regarded as a map $\Delta^{n-1} \to X$.

<u>claim</u>: $\partial_n \partial_{n+1} = 0 \quad (\partial^2 = 0)$

(same proof as before works)

so the singular homology groups form a chain complex

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

<u>Defn</u> the singular homology groups $H_n(X) = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$.

<u>observations</u>

- homeomorphic spaces have isomorphic singular homology groups.
- $C_n(X)$ not finite dimensional in general (in fact usually uncountable dim), so not clear that $H_n(X)$ is finitely generated, or $H_n(X) = 0$ if $n > \dim X$.
- <u>cycles</u>: chains with $\partial = 0$ (i.e. in $\ker(\partial)$).

$\xi = \sum n_i \sigma_i$

write this as $\xi = \sum \epsilon_i \sigma_i$ with $\epsilon_i = \pm 1$ (allowing repetition in the σ_i)

$\partial \xi$: sum of singular $(n-1)$-simplices with signs, there may be <u>cancelling pairs</u>, i.e. the same simplex map $\sigma : \Delta^{n-1} \to X$ but with opposite signs.

choose a maximal collection of cancelling pairs, and construct an n-dim complex K_ξ from a disjoint union of n-simplices Δ^n_i, one for each σ_i, and identify pairs of faces coming from cancelling pairs.

K_ξ is a manifold, except possibly on a subcomplex of dim $(n-2)$

(fact: actually $n-3$). can orient simplices using signs ϵ_i.

$H_1(X)$: cycles represented by maps of collections of oriented loops.

$H_2(X)$: cycles represented by maps of closed oriented surfaces.

a cycle in $H_1(X)$ represents 0 iff it bounds an oriented surface.

<u>Warning</u>: cycles are not necessarily manifolds in higher dimensions.

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<u>Prop^n</u> let X have path components X_α, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$

<u>Proof</u> $\sigma: \Delta^n \longrightarrow X$

$\uparrow$

path-connected $\Rightarrow$ image $\sigma(\Delta^n)$ also path connected

so $C_n(X)$ is a direct sum of $C_n(X_\alpha)$. the boundary maps preserve this direct sum decomposition, so $\ker(\partial_n)$, $\operatorname{im}(\partial_n)$, split into direct sums across path components, so $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$ □

<u>Prop^n</u> X non-empty, path connected, then $H_0(X) \cong \mathbb{Z}$, so for any X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path component.

<u>Proof</u> $H_0(X) = C_0(X)/\operatorname{im}(\partial_1)$ $\quad$ (as $C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0 = 0} 0$)

define a homomorphism $\epsilon: C_0(X) \rightarrow \mathbb{Z}$

$\sum n_i \sigma_i \longmapsto \sum n_i$ $\quad$ (surjective if $X \neq \phi$)

<u>claim</u> $\ker \epsilon = \operatorname{im} \partial_1$ if X path connected.

proof of claim:

- $\operatorname{im} \partial_1 \subset \ker \epsilon$: $\quad$ spose $\sigma: \Delta^1 \rightarrow X$, then $\epsilon\partial_1(\sigma) = \epsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]})$

$= 1 - 1 = 0$ □

- $\ker \epsilon \subset \operatorname{im} \partial_1$: suppose $\epsilon(\sum n_i \sigma_i) = 0$ $\{\Leftrightarrow \sum n_i = 0\}$

σ_i are 0-simplices. choose $\tau_i : I \longrightarrow X$ from same basepoint $x_0 \in X$ to $\sigma_i(v_0)$. So $\partial \tau_i = \sigma_i - \sigma_0$, where $\sigma_0 : [v_0] \longmapsto x_0$

so $\partial(\sum n_i \tau_i) = \sum n_i \sigma_i - \underbrace{\sum n_i \sigma_0}_{=0} = \sum n_i \sigma_i$ as required. $\square$.

so: $C_0(X)/\operatorname{im}(\partial_1) \xrightarrow{\epsilon} \mathbb{Z}$ and $\ker \epsilon = \operatorname{im}(\partial_1) \Rightarrow C_0(X) \cong \mathbb{Z}$ $\square$.

<u>Prop$^{\underline{n}}$</u> If $X = \{pt\}$ then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.

<u>Proof</u> there is a unique $\sigma_n : \Delta^n \longrightarrow X$ for each n.

$$\partial_n(\sigma_n) = \sum_i (-1)^i \sigma_{n-1} \leftarrow (n+1)\text{-terms. so } \begin{array}{l} \partial_n = 0 \ (n \text{ odd}) \\ \partial_n = \sigma_{n-1} \ (n \text{ even}) \end{array}$$

chain complex:

$$\begin{tikzcd}
\cdots \arrow[r] & C_3(X) \arrow[r] & C_2(X) \arrow[r] & C_1(X) \arrow[r] & C_0(X) \arrow[r] & 0 \\
\arrow[r, "\cong"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "\cong"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r] & 0 \\
\cdots & & H_2(X) \cong 0 \arrow[u] & H_1(X) \cong 0 \arrow[u] & H_0(X) \cong \mathbb{Z}. \arrow[u] &
\end{tikzcd}$$

so $H_n(\{pt\}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$ $\square$.