form the quotient space $X = \bigsqcup \Delta^n_\alpha / \sim$

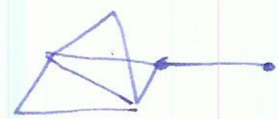
by identifying all faces in each set F_i by the canonical linear homomorphisms between them.

<u>Example</u> Δ^n_α : $[a_0, a_1, a_2]$ $\quad$ F_i : $F_1 = \{ [a_0, a_1], [b_2, b_3], [c_1, c_3] \}$

$[b_0, b_1, b_2]$ $\quad$ $F_2 = \{ [c_0], [d_0] \}$.

$[c_0, c_1, c_2]$

$[d_0, d_1]$ $\quad$ $X =$



<u>Remarks</u> • a 1-dim simplicial complex is an oriented graph.

• face identifications preserve orderings, so no two points in the interior of any face are identified, so $X =$ union of <u>open simplices</u> e^n_α, with induced maps $\sigma_\alpha : \Delta^n \to X$, so each Δ-complex is also a cell complex / CW complex, so

simplicial complexes $\subset$ Δ-complexes $\subset$ cell complexes.

<u>Simplicial homology</u>

X : Δ-complex

let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e^n_α of X. (i.e. formal sums of n-simplices)

an element of $\Delta_n(X)$ looks like $\sum_\alpha n_\alpha e^n_\alpha$ or $\sum_\alpha n_\alpha \sigma_\alpha$

$n_\alpha \in \mathbb{Z}$ $\quad$ $\sigma_\alpha : \Delta^n \to X$ characteristic map.

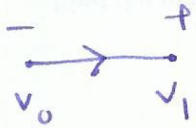
The boundary of an n-simplex is a collection of $(n-1)$-simplices

notation : $[v_0, \dots, \hat{v}_i, \dots, v_n]$ means $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$

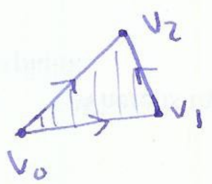
We define a <u>boundary map</u> ∂ which takes an n-simplex $[v_0, \dots, v_n]$ to its boundary :

$$\partial [v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \dots, \hat{v}_i, \dots v_n]$$

<u>Examples</u>

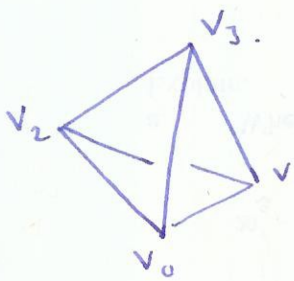


$$\partial [v_0, v_1] = [v_1] - [v_0]$$



$$\partial [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

("  ") .



$$\partial [v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2] .$$

so $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

<u>Lemma</u>

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

$$\partial_{n-1} \partial_n = 0 \quad (\text{or } \partial^2 = 0) .$$

<u>Intuition</u> : boundary of something is closed (has no boundary) .

Proof $\sigma : [v_0, \dots, v_n] \to X$

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]}$$

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^j (-1)^i \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots v_n]}.$$

$$+ \sum_{j>i} (-1)^{j-1} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}$$

↑ cancel in pairs, $\partial_{n-1}\partial_n(\sigma) = 0$. □

We now have a sequence of homomorphisms of abelian groups.

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $\partial_n \partial_{n+1} = 0$ for all n.

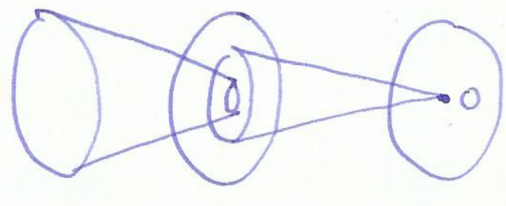
Defn This is called a <u>chain complex</u>

$$\partial_n \partial_{n+1} = 0 \implies \operatorname{Im}(\partial_{n+1}) \subset \ker(\partial_n)$$

Defn the <u>n-th homology group</u> of the chain complex is

$$H_n = \ker(\partial_n) / \operatorname{im}(\partial_{n+1})$$

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$



elements of $\ker(\partial_n)$ are <u>cycles</u> (n-cycles)

elements of $\mathrm{im}(\partial_{n+1})$ are <u>boundaries</u> (($n+1$)-boundaries.)

elements of the abelian group H_n are cosets of $\mathrm{Im}(\partial_{n+1})$ and are called <u>homology classes</u>. Two cycles representing the same homology class are said to be <u>homologous</u> (i.e. their difference is a boundary).

When $C_n = \Delta_n(X)$ the homology group $H_n^{\Delta}(X)$ is the <u>n-th simplicial homology group</u> of X

<u>Examples</u>

① point $X = [v_0]$

$$\cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$

$$\cdots \to 0 \to 0 \xrightarrow[\partial_1 = 0]{} \mathbb{Z} \xrightarrow[\partial_0 = 0]{} 0$$

$$H_n^{\Delta}(X) = 0 \qquad n \geq 1$$

$$H_0^{\Delta}(X) \cong \mathbb{Z} \cong \ker(\partial_0) / \mathrm{im}(\partial_1).$$

② edge $X = [v_0, v_1]$

$$\cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$

$$0 \to \mathbb{Z} \xrightarrow[\partial_1]{} \mathbb{Z}^2 \to 0$$

$$\partial_1([v_0, v_1]) = [v_1] - [v_0]$$

primitive in $\mathbb{Z}^2$ generated by $[v_0], [v_1]$.

$\ker(\partial_1) = 0$

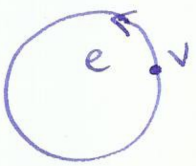
$\Rightarrow H_1^{\Delta}(X) \cong 0$

- $H_0^\Delta(X) = \ker(0) / \mathrm{im}(\partial_1) = \langle [v_0], [v_1] \rangle / \langle [v_1] - [v_0] \rangle \cong \langle [v_1]-[v_0], [v_1] \rangle / \langle [v_1]-[v_0] \rangle \cong \mathbb{Z}$

- $H_n^\Delta(X) = 0 \quad n \geq 1$

  $H_0^\Delta(X) \cong \mathbb{Z}$.

③ $X = S^1$ $\quad$ Δ-complex: $[v_0, v_1]$, $\{ \{[v_0], [v_1]\} \}$.



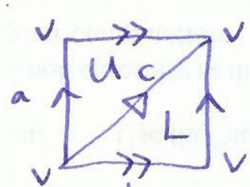
$\partial(e) = v - v = 0$

$$\cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$
$$0 \qquad \mathbb{Z} \xrightarrow[\partial_1 = 0]{} \mathbb{Z}$$

$H_n^\Delta(X) = 0 \quad n \geq 2$

$H_n^\Delta(X) \cong \mathbb{Z} \quad n = 0, 1$.

④ $X = T^2$



1 vertex v

3 edges a, b, c

2 faces U, L

$$\to \Delta_3(X) \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$
$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow[\partial_2]{} \mathbb{Z}^3 \xrightarrow[\partial_1]{} \mathbb{Z} \longrightarrow 0$$

$$\left.\begin{aligned} \partial_1(a) &= v - v = 0 \\ \partial_1(b) &= v - v = 0 \\ \partial_2(c) &= v - v = 0 \end{aligned}\right\} \partial_1 = 0 \implies H_0^\Delta(T) \cong \mathbb{Z}.$$