Topology I Math 70800 (second semester)

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Text: Algebraic Topology, Allen Hatcher (available online)

§2 Homology

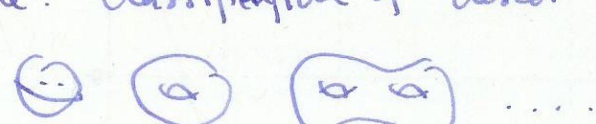
Motivation: topological space X (metric space) (Cayley graph of group) ← • classification: not feasible, too many
• invariants: help us distinguish spaces which are different

algebraic topology: algebraic invariants of topological spaces.

Example Fundamental group X space ⇝ $\pi_1 X$ group fundamental group of X.

good: distinguishes many spaces in practice, easy to define.

key example: classification of closed orientable surfaces



S^2 T^2 S_2

$\pi_1 X$: $\{1\}$ $\mathbb{Z}^2$ $\langle a,b,c,d \mid [a,b][c,d] \rangle$ etc.

bad: • computations may be difficult (e.g. X finite cell complex ⇝ $\pi_1 X$ ⇐ gives group presentation, but no algorithm to decide
– if trivial group
– if any element trivial)

• "only sees 2-skeleton"

• $\pi_1(S^n) = \{1\}$ for all n.

remark: higher homotopy groups $\pi_k X$, built from homotopy classes of maps $S^k \to X$

good: distinguishes spheres.

bad: hard to compute (even for spheres!) e.g. $\pi_3 S^2 \cong \mathbb{Z}$.
(Hopf fibration...).

Homology bad: more work to define

good: n-dimensional versions depends only on $(n+1)$-skeleton.

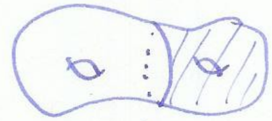
$H_n(X)$ abelian group, often computable in practice, with many computational techniques.

Idea recall: $\pi_1 X$: (non-abelian) group of loops based at $x_0 \in X$, up to homotopy

want $H_1 X$: abelian group of (formal sums) of loops (not necessarily based at x_0) up to equivalence: bounding 2d subsets of X.

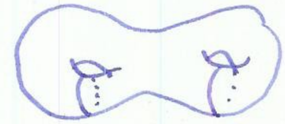
Example

nontrivial element of $H_1 X$



trivial/zero in $H_1 X$.

formal sum:



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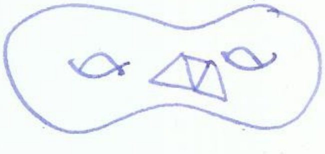
same element of $H_1 X$.

to do: · decide what the loops can be
· define equivalence
· show how to compute things in useful examples

bonus: · can compute H_n from cell-structure info.

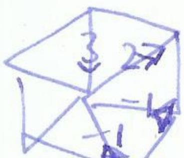
<u>sketch</u>: we can build a group from (say) a simplicial structure on X



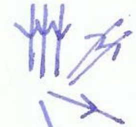
← triangulate S

- take formal sums of edges of the simplices, call this ~~$C_1(S)$~~ C_1 (~~vector space~~ w/ basis edges) abelian group

<u>Example</u>

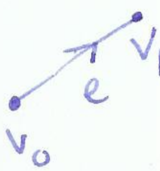


imagine:



<u>Q</u>: how do we tell if such a formal sum corresponds to a collection of closed curves?

<u>A</u>: there is a <u>boundary map</u> ∂: which takes an oriented edge e

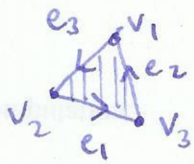


to its (oriented) boundary $v_1 - v_0 \in C_0$ ← formal sums of vertices. abelian gp (~~vector space~~ with basis vertices)

i.e. $\partial : C_1 \to C_0$

<u>closed loops go to zero under ∂</u>

<u>Example</u>



$$\partial e_1 = v_3 - v_2$$
$$\partial e_2 = v_1 - v_3$$
$$\partial e_3 = v_2 - v_1$$

so $\partial(e_1 + e_2 + e_3) = 0$

<u>equivalence</u>: let C_2 be the formal sums of triangles: $C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$

$\partial f = e_1 + e_2 + e_3$.

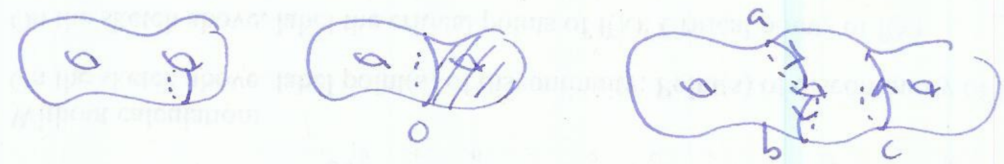
<u>Q</u>: is this useful? note:

$$\partial f_1 = e_1 + e_2 - e_3$$
$$\partial f_2 = e_3 + e_4 - e_5$$

$$\partial(f_1 + f_2) = e_1 + e_2 - \cancel{e_3} + \cancel{e_3} + e_4 - e_5$$
$$= e_1 + e_2 + e_4 - e_5.$$

$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ define $H_1 = \ker(\partial_1) / \mathrm{im}\,\partial_2$ abelian gp.

$H_1 =$ "things with no boundary" / "things which bound".

Examples 

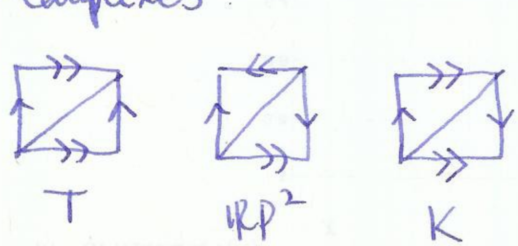
$a+b+c=0$

or $c = \cancel{a+b}$ $-a-b$.

§2.1 Simplicial homology

~~Recall~~ Δ-complexes

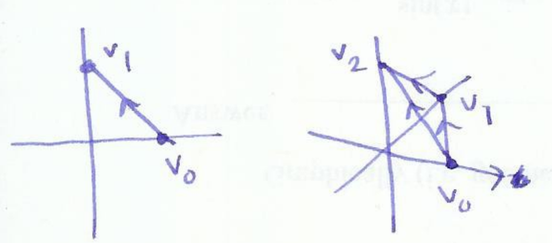
simplicial complex: closed simplices homeomorphically embedded

Examples 

Δ-complex: union of interiors, but closures need not be embedded.

Def^n n-simplex: small convex set in $\mathbb{R}^n$ ~~containing~~ consisting of the convex hull of $n+1$ points $v_0, \dots, v_n$, which do not lie in a hyperplane of dim$<n$ (equivalently, the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent) the points v_i are the vertices of the simplex.

Example "standard" n-simplex, spanned by unit vectors in $\mathbb{R}^{n+1}$



or: $\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \}$

important point: for homology, n-simplex means "n-simplex, together with an order for the vertices" notation: $[v_0, v_1, \dots, v_n]$

e.g: this determines orientations on each edge according to increasing subscript: $[v_i, v_j]$ $v_i \longrightarrow v_j$ $i<j$.

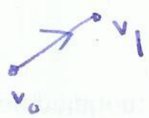
the ordering determines a canonical linear homeomorphism from the standard simplex to $[v_0, \dots, v_n]$ sending

$$(t_0, \dots, t_n) \longmapsto \sum_i t_i v_i$$

the coefficients t_i are called <u>barycentric coordinates</u> on $[v_0, \dots, v_n]$

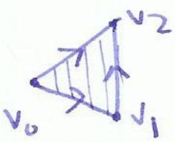
<u>Def^n</u> A <u>face</u> of a simplex $[v_0, \dots, v_n]$ is the subsimplex with vertices consisting of any non-empty subset of the v_i, ordered by the induced order from $[v_0, \dots, v_n]$

<u>Examples</u>



$[v_0, v_1]$

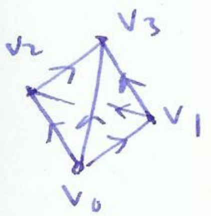
$[v_0], [v_1]$



$[v_0, v_1, v_2]$

$[v_0, v_1], [v_0, v_2], [v_1, v_2]$

$[v_0], [v_1], [v_2]$



$[v_0, v_1, v_2, v_3]$

$[v_0, v_1, v_2]$ $[v_0, v_1, v_3]$ $[v_0, v_2, v_3]$ $[v_1, v_2, v_3]$

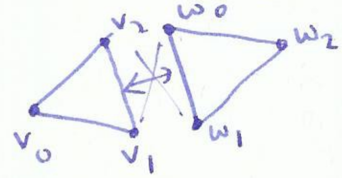
$[v_0, v_1]$ $[v_0, v_2]$ $[v_0, v_3]$ $[v_1, v_2]$ $[v_1, v_3]$ $[v_2, v_3]$

$[v_0]$ $[v_1]$ $[v_2]$ $[v_3]$.

etc.

<u>Def^n</u> A <u>Δ-complex</u> is the quotient space of a collection of disjoint simplices, obtained by identifying some subset of their faces, by the canonical order-preserving linear maps between them.

<u>Warning</u>:



not allowed! can't glue $[v_1, v_2]$ to $[w_1, w_0]$ <u>only</u> to $[w_0, w_1]$

<u>more formally</u>: let Δ^n_α be a collection of simplices of various dimensions.

let F_i be a set of faces of the Δ^n_α, of the same dimension