# Review questions

(some of these are hard)

1. Write down the infima and suprema of the following sets. In each case, state whether the inf and sup are minimum or maximum values.
   (a) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$; (b) $\{\frac{3n}{4n+1} : n \in \mathbb{N}\}$; (c) $\mathbb{Q} \cap (0, 1)$; (d) $\{x \in \mathbb{Q} : x^2 < 7\}$; (e) $\{\sin n : n \in \mathbb{Z}^+\}$
2. In the following, $X$ and $Y$ are nonempty sets of real numbers. Using the definition of sup, prove that
   (a) $\sup\{\frac{3n-1}{4n} : n \in \mathbb{Z}^+\} = \frac{3}{4}$
   (b) $\sup\{\cos \frac{1}{n} + (-1)^n : n \in \mathbb{Z}^+\} = 2$
   (c) $\sup(X \cup Y) = \max\{\sup X, \sup Y\}$
   (d) Let $a > 0$. Then $\sup(aS) = a \sup(S)$, where $aS = \{ax : x \in S\}$
   (e) $\sup(X + Y) = \sup X + \sup Y$, where $X + Y = \{x + y : x \in X, y \in Y\}$
   (f) Find a counterexample to show that $\sup(XY) = \sup X \sup Y$ is generally false, where $XY = \{xy : x \in X, y \in Y\}$
3. Let $f(x)$ and $g(x)$ be functions $: \mathbb{R} \to \mathbb{R}$. Suppose that $f(x) \leq g(x)\ \forall x \in [a, b]$. Prove that $\sup\{f(x) : x \in [a, b]\} \leq \sup\{g(x) : x \in [a, b]\}$
4. Find the following limits and prove your result:
   (a) $\lim\limits_{n\to\infty} \dfrac{n}{3n+4}$; (b) $\lim\limits_{n\to\infty} \sqrt{(n^2 + 6n)} - n$; (c) $\lim\limits_{n\to\infty} \dfrac{2n + \sin n}{n + 2}$
5. Show that the sequence $\left\{\dfrac{2n+1}{n}\right\}$ does not converge to 1.
6. Give an example of a bounded sequence that does not converge.
7. Suppose $\{s_n\}$ and $\{t_n\}$ are sequences such that $\{s_n\}$ and $\{s_n + t_n\}$ converge. Prove that $\{t_n\}$ converges.
8. Let $x_n \to x$ and $y_n \to y$. Show from the definition of convergence that $3x_n - 2y_n \to 3x - 2y$
9. Let $x_n$ be a bounded sequence and let $y_n \to 0$. Show that $x_n y_n \to 0$.
10. Prove that if $x_n \to x$, then $|x_n| \to |x|$.
11. Show that if $x_n$ is a convergent sequence, then $x_{n+1} - x_n \to 0$. Is the converse true? Prove or disprove.
12. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\lim_{n\to\infty} b_n = 0$ where $b_n \neq 0$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$.
13. Consider the sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n\} \to 0$. Must $\{a_n b_n\} \to 0$? Prove or disprove.
14. Consider the sequence $\{a_n\}$, where $a_n = \dfrac{1}{4n^2 - 1}$. Define the sequence $\{s_n\}$ by $s_n = a_1 + a_2 + \cdots + a_n$. Determine whether or not $\{s_n\}$ converges. If so, find the limit.
15. Is it possible to have an unbounded sequence $\{a_n\}$ so that $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = 0$? Explain.
16. Consider the sequences $\{a_n\}$ and $\{b_n\}$, where $b_n = \dfrac{a_n + 1}{a_n - 1}$. If $\{b_n\} \to 0$, prove that $\{a_n\} \to -1$.
17. If the sequence $\{a_n\} \to +\infty$, and $\alpha, \beta$, and $k$ are positive constants, prove that $\dfrac{\alpha a_n}{k + \beta b_n} \to \dfrac{\alpha}{\beta}$.
18. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms such that $\{\dfrac{a_n}{b_n}\} \to 0$. Prove that if $\{a_n\} \to +\infty$, then so does $b_n$.
19. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms such that $\{\dfrac{a_n}{b_n}\} \to 0$. Prove that if $\{b_n\}$ is bounded, then $a_n \to 0$.
20. Prove that the sequence $a_n = \sin(\frac{n\pi}{2})$ diverges, and find all the subsequential limits.

21. Let the sequence $\{a_n\}$ be defined by $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $\{a_n\}$ is unbounded by showing that there exists some subsequence that is unbounded.

22. Let $\{s_n\}$ be a sequence of real numbers. Suppose that $\liminf_{n\to\infty} s_n = +\infty$. Show that $\lim_{n\to\infty} s_n = +\infty$.

23. Show directly from the definition of convergence that $\displaystyle\lim_{n\to\infty} \frac{4n^2}{3n^2+1} = \frac{4}{3}$

24. Show that the intersection of a finite number of open subsets of $\mathbb{R}$ is open.

25. Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in $\mathbb{R}$. Recall that for all $x, y \in \mathbb{R}$, $d(x,y) = |x-y|$ is the distance between $x$ and $y$. Prove that $\{d(p_n, q_n)\}$ is a convergent sequence using the following outline:
    (a) Show that $d(x_1, y_1) \leq d(x_1, x_2) + d(x_2, y_2) + d(y_2, y_1)$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$
    (b) Use part (a) to demonstrate that $|d(p_n, q_n) - d(p_m, q_m)| \leq d(q_m, q_n) + d(p_n, p_m)$
    (c) Deduce that $\{d(p_n, q_n)\}$ is a Cauchy sequence
    (d) Conclude that $\{d(p_n, q_n)\}$ must converge.

    Answer with a single proof (or a couple of Lemmas + main proof) rather than just answers to the parts.

26. Prove *directly* that if $\{s_n\}$ and $\{t_n\}$ are Cauchy, so is $\{s_n t_n\}$ (do not use: in the reals, Cauchy $\Leftrightarrow$ convergent).

27. Give an example of a set with exactly 2 accumulation points.

28. Let $S$ be a nonempty set of reals that is bounded above, and let $x = \sup S$. Prove that either $x$ belongs to $S$ or $x$ is an accumulation point of $S$.

29. Show that the sequence defined by $a_1 = 1$ and $a_{n+1} = \frac{1}{3}(a_n + 1)$ for $n > 1$ is convergent, and find the limit.

30. Prove that every set of the form $\{x : a < x < b\}$ is open and every set of the form $\{x : a \leq x \leq b\}$ is closed

31. If $E \subseteq \mathbb{R}$ is bounded, prove that $\overline{E}$ is bounded.

32. Let $E \subseteq \mathbb{R}$. Prove that $\text{int}(E)$ is open, and that if $S$ is any open set contained in $E$, then $S \subseteq \text{int}(E)$.

33. Define $f : (-2, 0) \to \mathbb{R}$ by $f(x) = \frac{x^2-4}{x+2}$. Prove that $f$ has a limit at $-2$, and find it.

34. Give an example of a function $f : (0,1) \to \mathbb{R}$ that is bounded and has a limit at every point except $x_0 = 1$. Use the definition to justify the example.

35. Define $f : (0,1) \to \mathbb{R}$ by $f(x) = \cos(\frac{1}{x})$. Does $f$ have a limit at 0? Justify.

36. Define $f : (0,1) \to \mathbb{R}$ by $f(x) = x\cos(\frac{1}{x})$. Does $f$ have a limit at 0? Justify.

37. Show using the ($\epsilon$-$\delta$)-definition that $f(x) = 2x + 5$ is continuous at $x = 2$.

38. Show using the ($\epsilon$-$\delta$)-definition that $f(x) = \frac{1}{x+1}$ is continuous at $x = 1$.

39. Show using the ($\epsilon$-$\delta$)-definition that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

40. Show using the ($\epsilon$-$\delta$)-definition that $f(x) = mx + b$ is continuous on $\mathbb{R}$.

41. Show that if $f$ is continuous at $x_0$ then $|f|$ is continuous at $x_0$, in three ways: directly from the definition; using sequences, using composition of functions.

42. If $|f|$ is continuous is $f$ continuous?

43. Use the definition of the derivative to find the derivative of $f(x) = x^2 + 2$ at $x = 3$.

44. Use the definition of the derivative to find the derivative of $f(x) = 1/x$.

45. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$.

46. Show that $f(x) = |x|$ is not differentiable at $x = 0$.

47. Is $f(x) = x|x|$ continuous at $x = 0$? Is it differentiable at $x = 0$?