<u>proofs</u>: similar to proofs for sequences □.

why do we want x_0 accumulation point of intersection of domains?

consider $f(x)=\sqrt{x}$, $g(x)=\sqrt{-x}$, $\lim_{x\to 0} \sqrt{x}\cdot\sqrt{x}$?

<u>HW 5.2 Q1,5 ~~8~~, 11, 12, 15.</u>

Read: 5.2.4, 5.2.5, 5.2.6.

<u>Examples</u>

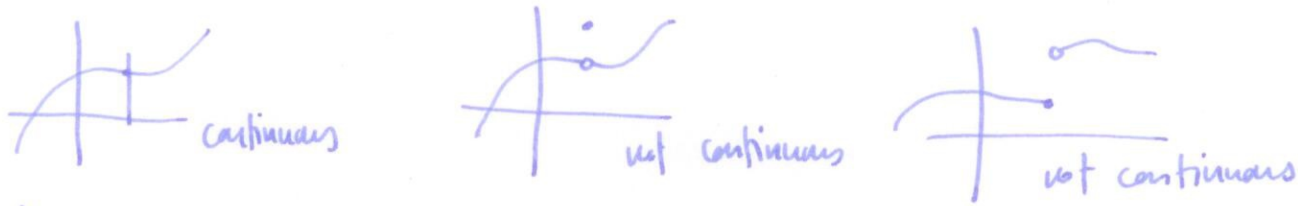
- Polynomials: $\lim_{x\to x_0} p(x) = p(x_0)$ ← follows from algebra of limits.
- Characteristic function: $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ $\quad \lim_{x\to x_0} \chi_{\mathbb{Q}}(x)$ DNE for all x_0
- Dirichlet function: $f(x) = \begin{cases} 0 & \text{if } x \text{ irrational or } x=0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$

$\lim_{x\to x_0} f(x) = 0$ ~~if x_0 irrational~~ for all x_0
~~x_0 rational.~~

note $\lim_{x\to x_0} f(x) = f(x)$ if x irrational ! so cts at irrationals
$\qquad\qquad\quad \neq f(x)$ if x rational $\quad$ not cts at rationals.

<u>§5.4 Continuity</u>

<u>Example</u>



<u>Def^n</u> (limit) Let f be defined in a neighbourhood of x_0. Then f is cts at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$

<u>Def^n</u> (ϵ-δ) Let f be defined in a nbd of x_0. Then f is cts at x_0 iff for every $\epsilon > 0$ there is a $\delta > 0$ s.t. if $|x-x_0|<\delta$ then $|f(x)-f(x_0)|<\epsilon$

Q: do you need $0 < |x - x_0|$ above?

Def$^{\underline{n}}$ (neighbourhood) Let f be defined in a nbd of x_0. Then f is cts at x_0 iff for every open neighbourhood V of $f(x_0)$, there is an open nbd U of x_0 s.t. $f(U) \subseteq V$.

Def$^{\underline{n}}$ (sequence) Let f be defined in a nbd of x_0. Then f is cts at x_0 iff for all sequences ~~of~~ $x_n \to x_0$, $f(x_n) \to f(x_0)$.

Th$^{\underline{m}}$ These definitions are equivalent.

Proof HW. $\square$

Remark in def$^{\underline{n}}$ of limit $\lim_{x \to x_0} f(x)$, we do not care about $f(x_0)$, but for continuity we do!

Discontinuity

Def$^{\underline{n}}$ A function f is ~~cts~~ discontinuous at a if it is not continuous at a.

Negate definition of cts. (e.g. (ϵ-δ) def$^{\underline{n}}$):

f is discontinuous at a iff there is an $\epsilon > 0$ s.t. for every $\delta > 0$ there is an x with $|f(x) - f(a)| > \epsilon$ and $|x - a| < \delta$.

Remarks: How can f not be cts at a?

- f not defined at a  
- $\lim_{x \to a} f(x)$ DNE.
- $\lim_{x \to a} f(x) \neq f(a)$.

How to show f discontinuous:

- find $\epsilon > 0$ and $x_n \to a$ s.t. $|f(x_n) - f(a)| > \epsilon$.
- find two sequence $x_n \to a$ $y_n \to a$ st. $\lim f(x_n) \neq \lim f(y_n)$
- find sequence $x_n \to a$ s.t. $\lim f(x_n) \neq f(a)$.

Examples $f(x) = 2x+3$: show cts using each defn.

$f(x) = \frac{1}{x}$ on $(0,\infty)$.

Examples of continuous functions

HW 5.7 7,9,11.

- polynomials
- $\sin(x), \cos(x)$
- $\tan^{-1}(x)$
- exponentials
- logarithms.

§5.5 Properties of continuous functions

Thm Let $f,g : E \to \mathbb{R}$, and let $c \in E$, with f,g cts at $x_0 \in E$. Then

- cf is cts at x_0
- $f+g$ cts at x_0
- fg cts at x_0
- f/g cts at x_0 provided $g(x_0) \neq 0$.

Proof for all $\epsilon > 0$ there is δ_1 s.t. if $|x-x_0| < \delta_1$, then $|f(x)-L_1| < \epsilon$

δ_2 $|x-x_0| < \delta_2$ $|g(x)-L_2| < \epsilon$.

$$|cf(x) - cL_1| = c\,|f(x)-L_1| \le c\epsilon$$

$$|f(x)+g(x)-L_1-L_2| \le |f(x)-L_1| + |g(x)-L_2| \le 2\epsilon.$$

$$|f(x)g(x) - L_1L_2| = |f(x)g(x) - L_1 g(x) + L_1 g(x) - L_1L_2|$$

$$\le |g(x)|\,|f(x)-L_1| + |L_1|\,|g(x)-L_2|.$$

$$\left|\frac{f(x)}{g(x)} - \frac{L_1}{L_2}\right| = \left|\frac{f(x)}{g(x)} - \frac{L_1}{g(x)} + \frac{L_1}{g(x)} - \frac{L_1}{L_2}\right| \le \frac{1}{|g(x)|}|f(x)-L_1| + \frac{|L_1|}{|g(x)-L_2|}. \quad \square$$

Corollary • every polynomial is cts on $\mathbb{R}$

• every rational function is cts on its domain.

(i.e. $f(x)/g(x)$ cts when $g(x) \neq 0$).

Thm Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ s.t. $f(A) \subset B$.
$A \subset \mathbb{R}$, $B \subset \mathbb{R}$

If f is cts at x_0 and g is cts at $y_0 = f(x_0)$, then $g \circ f$ is cts at x_0.

Proof for all $\epsilon > 0$ there is $\delta_1 > 0$ s.t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \epsilon$
$\delta_2 > 0$ $\quad |y - y_0| < \delta_2 \Rightarrow |g(y) - g(y_0)| < \epsilon$.

choose $\epsilon = \delta_2$:
$$|x - x_0| < \delta_1 \Rightarrow |\underbrace{f(x)}_{y} - \underbrace{f(x_0)}_{y_0}| < \delta_2$$
$$|f(x) - f(x_0)| < \delta_2 \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon . \qquad \Box.$$

Alternate proof (sequences)

f cts means $\quad x \to x_0 \Rightarrow f(x) \to f(x_0)$
g cts $\quad y \to y_0 \Rightarrow g(y) \to g(y_0)$

so
$$g(f(x)) \to g(f(x_0)). \qquad \Box.$$

HW: 5.5. 1, 2, 3, 4
5.6 8, 9, 12.

§5.6 Uniform continuity

Defn $f : A \to \mathbb{R}$ is cts on A if for every $a \in A$, and every $\epsilon > 0$, there is a $\delta(a, \epsilon)$ s.t. if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Defn $f : A \to \mathbb{R}$ is uniformly continuous if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ s.t. for all $a \in A$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Example $f(x) = x^2$ cts with $\delta = \min\{1, \frac{\epsilon}{1 + 2|a|}\}$ not uniformly cts on $\mathbb{R}$, but uniformly cts on $[0,1]$.

Not uc: $1/x$, $\sin(1/x)$ $\quad$ uc: $\sin(x)$, $\sqrt[3]{x}$. $\quad |\sqrt{x+h} - \sqrt{x}| = \left| \frac{h}{\sqrt{x+h} + \sqrt{x}} \right| \leq \left| \frac{h}{\sqrt{h}} \right|$.

To show not uniformly cts: negation of def$^{\underline{n}}$.

Find (x_n) (y_n) s.t. $|x_n - y_n| \to 0$ but $|x_n - y_n| \geq \epsilon$.

Thm If f uniformly cts on a bounded set, then f bounded. (note: need not be closed!)

Thm If f is cts on $[a,b]$ then it is uniformly cts on $[a,b]$.

Proof Suppose not, then there is (x_n) (y_n) s.t. $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon$. [Bolzano-Weierstrass] (x_n) has a convergent subsequence, converging to x_0 say. By f cts $f(x_n) \to f(x_0)$

$|x_0 - y_n| \leq \underbrace{|x_0 - x_n|}_{\to 0} + \underbrace{|x_n - y_n|}_{\to 0}$ by cts $f(y_n) \to f(x_0)$

$\Rightarrow |f(x_n) - f(y_n)| \to 0$ $\#$. D.

§7.2 The derivative

(I interval in $\mathbb{R}$)

Def$^{\underline{n}}$ $f: I \to \mathbb{R}$ $x_0 \in I$ The derivative of f at x is

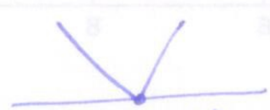
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ provided this limit exists.

Def$^{\underline{n}}$ If $f'(x_0)$ exists then we say f is differentiable at x_0.

Def$^{\underline{n}}$ If f is differentiable for all $x_0 \in I$, then say f is differentiable on I.

$\chi_{\mathbb{Q}}$ not differentiable anywhere.

Example



Example find $f'(x_0)$ if $f(x) = x^2$

$$f'(x_0) = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} x + x_0 = 2x_0.$$

Thm Differentiable $\Rightarrow$ continuous.