Def$^{\underline{n}}$ Let $f: E \to \mathbb{R}$, x_0 accum point of $E \cap (x_0, \infty)$

then we say $\lim_{x \to x_0+} f(x) = \infty$ iff for every $M > 0$ there is a $\delta > 0$ s.t.

for all $x \in E$ with $x_0 < x < x_0 + \delta$, $f(x) \geq M$.

Similarly - define $\lim_{x \to x_0+} f(x) = -\infty$

- left hand limits
- two sided limits — note $\lim_{x \to x_0} f(x) = \pm\infty$ still DNE!
- define $\left.\begin{array}{l} \lim_{x \to \infty} f(x) \\ \lim_{x \to -\infty} f(x) \end{array}\right\}$ see HW.

Examples - find and prove $\lim_{x \to 1} \dfrac{x^2 - x - 2}{2x - 2}$.

- f, g defined on $(a-c, a+c) \setminus \{a\}$. If $\lim_{x \to a} f(x) = L > 0$

and $\lim_{x \to a} g(x) = +\infty$, then $\lim_{x \to a} f(x) g(x) = +\infty$.

Proof $f(x) > L/2$, $g(x) > 2M/L$.

§5.2 Properties of Limits

Uniqueness

Th$^{\underline{m}}$ suppose $\lim_{x \to x_0} f(x) = L$, then L is unique.

(i.e. if $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} f(x) = L_2$ then $L_1 = L_2$).

Proof can use sequence def$^{\underline{n}}$.

Boundedness $f : E \subseteq \mathbb{R} \to \mathbb{R}$

Thm Suppose $\lim_{x \to x_0} f(x) = L$, then there is an interval $(x_0 - c, x_0 + c)$ and $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in (x_0 - c, x_0 + c) \cap E$.

Proof Use δ for $\epsilon = 1$. If $x_0 \notin E$ then $M = |L| + 1$
$x_0 \in E$ then $M = |L| + 1 + |f(x_0)|$ $\square$.

Boundedness away from zero

Thm If $\lim_{x \to x_0} f(x) = L \neq 0$, then there is an interval $(x_0 - c, x_0 + c)$ and $m > 0$ s.t. $f(x) \geq m > 0$ for all $x \in (x_0 - c, x_0 + c) \cap E$.

Proof: HW 5.2.5.

Algebra of limits.

Example $\displaystyle \lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1} = \frac{\lim_{x \to 0} x^2 - 1}{\lim_{x \to 0} x^2 + 1} = \frac{\left(\lim_{x \to 0} x^2\right) - 1}{\left(\lim_{x \to 0} x^2\right) + 1}$ etc.

Scalar multiple:
$$\lim_{x \to x_0} k f(x) = k \lim_{x \to x_0} f(x)$$

sums:
$$\lim_{x \to x_0} f(x) + g(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) \quad \text{(domains!)}$$

products:
$$\lim_{x \to x_0} f(x) g(x) = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x)$$

quotients:
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} \quad \text{if } \lim_{x \to x_0} g(x) \neq 0.$$