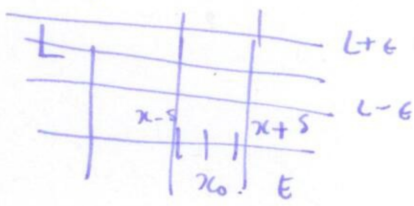
s.t. for every $x \in E$, if $0<|x-x_0|<\delta$ then $|f(x)-L|<\epsilon$.

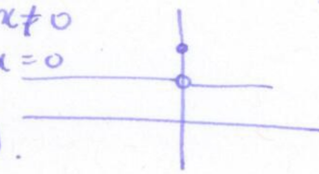


Note: in sequence convergence $N(\epsilon)$, here $\delta(\epsilon)$.

• why $0<|x-x_0|<\delta$
$\uparrow$
$\leq$?

Example $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$

$\lim_{x\to 0} f(x) = 1 \ (\neq 2)$.



Examples

• prove: $\lim_{x\to 3} 3x-1 = 8$ rough work: consider $f(\overset{x}{3+\delta}) = 3(\overset{x}{3+\delta})-1$

$|3(\overset{x}{3+\delta})-1-8| = |9+3\delta-9| = |3\delta| < \epsilon$.

choose $\delta < \epsilon/3$

proof: given $\epsilon > 0$, choose $\delta < \frac{\epsilon}{3}$.

then if $|3-x|<\delta$ then $|f(x)-L| = |3x-1-8| = |3x-9| = 3|x-3| < 3\delta < \epsilon$, as required $\square$.

• prove: $\lim_{x\to 2} x^2 = 4$ rough work: $|(\overset{x}{2+\delta})^2-4| = |4+2\delta+\delta^2-4| = |2\delta+\delta^2| = \delta|2+\delta|$

note: if $|\delta|<1$ then $|2+\delta|<3$, so $\leq 3\delta$.

proof: given $\epsilon>0$, choose $\delta < \epsilon/3$.

let $|2-x|<\delta$, then $|f(x)-L| = |x^2-4| = |(x+2)(x-2)| \leq \underbrace{|x+2|}_{\leq 3}\underbrace{|x-2|}_{\leq \delta} \leq 3\delta < \epsilon$.

for $\delta \leq 1$ $\square$.

Def^{n} (sequential definition). Suppose $f: E \subseteq \mathbb{R} \to \mathbb{R}$ is a function with domain E, and suppose x_0 is an accumulation point of E. Then $\lim_{x\to x_0} f(x) = L$ iff for every sequence (x_n) in E ($x_n \neq x_0$ for all n) converging to x_0, the sequence $(f(x_n))$ converges to L, (i.e. $\lim_{n\to\infty} f(x_n) = L$).

~~Proof~~ Thm. (ϵ-δ)-limit def^{n} equivalent to sequential limit def^{n}.

Examples • $f(x) = 3x-1$, $\lim_{x\to 1} f(x) = 2$. Pick sequence $x_n \to 1$ and calculate $\lim f(x_n)$.

• $f(x) = \sin(\frac{1}{x})$. Q: does $\lim_{x\to 0} f(x)$ exist? (pick sequence with $f(x_n)$ divergent).

• $f(x) = \begin{cases} 2x+1 & \text{if } x \text{ rational} \\ 5x-8 & \text{if } x \text{ irrational} \end{cases}$ show $\lim_{x\to 3} f(x) = 7$, and no other limits exist.

• $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ $f(x) = 2x+1$ $\lim_{x\to 0} f(x)$ $\lim_{x\to 1} f(x)$?

HW: 5.1 Q 1, 3, 10 (don't use sequences), 13, 16, 17

Equivalence of convergence definitions

$\lim_{x \to x_0} f(x) = L$
- for all $\epsilon > 0$, there is a $\delta > 0$ s.t. for all $|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$
- for every sequence $x_n \to x_0$ in $E \setminus x_0$, $f(x_n) \to L$.

Thm: These two definitions are equivalent

Proof $\Rightarrow$ $x_n \to x_0$ means for every $\delta > 0$ there is an N s.t. $|x_n - x_0| < \delta$ for all $n \geq N$

but then $|f(x_n) - L| < \epsilon$ for all $n \geq N$, so $f(x_n) \to L$.

$\Leftarrow$ contrapositive: suppose ϵ-δ-limit not L.

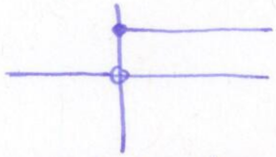
negation: There is an $\epsilon > 0$ s.t. for all $\delta > 0$ ~~with $|x$~~ there is an x with $|x - x_0| < \delta$ s.t. $|f(x) - L| \geq \epsilon$

pick $\delta = \frac{1}{n}$, there is x_n with $|x_n - x_0| < \delta = \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$.

consider the sequence (x_n). Then $x_n \to x_0$ but $f(x_n) \not\to L$. $\square$.

One sided limits

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



can think of right hand limit as ordinary limit of new function with domain restricted to right hand side. $f: E \subset \mathbb{R} \to \mathbb{R}$ for right hand limit at x_0 use $E \cap (x_0, \infty)$

left hand $E \cap (-\infty, x_0)$

Defn (ϵ-δ): $f: E \subset \mathbb{R} \to \mathbb{R}$ and let x_0 be an accumulation point of $E \cap (x_0, \infty)$

Then $\lim_{x \to x_0^+} f(x) = L$ iff for all $\epsilon > 0$ there is a $\delta > 0$ s.t. for all x with

$|x - x_0| < \delta$ ($x_0 < x < x_0 + \delta$!) we have $|f(x) - L| < \epsilon$.

Similarly for left hand limits and sequence defns.

HW 5.1 Q25, 28

Infinite limits

Consider $\lim_{x \to 0+} \frac{1}{x}$: DNE. But can say something more precise.