<u>Proof</u> $\Rightarrow$ E bounded $\Rightarrow$ any sequence in E bounded $\Rightarrow$ convergent subsequence.

$\Leftarrow$ suppose every sequence in E has a convergence subsequence (converges <u>in E</u>).

E bounded: spose not, construct unbounded monotonic sequence $\to \pm\infty$, no convergent subsequence #.

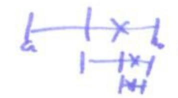
E closed: spose not, then there is an accumulation point x of E which does not lie in E. So for every $a > 0$, there are ∞ many points of E in $(x-a, x+a)$. So may choose a sequence x_n in E s.t. $x_n \to \overset{x}{a} \notin E$ #. $\square$.

<u>Corollary</u> $E \subseteq \mathbb{R}$ is closed and bounded iff every infinite subset of E has an accumulation point in E. (explain why need bounded, closed, infinite) ⊛

<u>Th^m^</u> Suppose that f is locally bounded at each point of a closed and bounded set E. Then f is bounded on E.

<u>Proof</u> suppose f not bounded, then there is a sequence $x_n \in E$ with $f(x_n) \to +\infty$. but (x_n) has a subsequence converging to $x \in E$. Local boundedness implies there is $a > 0$, M s.t. f bounded by M on $(x-a, x+a)$, but this interval contains ∞-many points of (x_n) # $\square$.

⊛ alternate proof: bounded implies contained in interval, $I = [a,b]$ cut in half, at least one half has infinitely many points, choose one a_1. Cut in half again, at least one half contains infinitely many points, choose $a_2 \neq a_1$. Continue, gives sequence (a_n) which



is <u>Cauchy</u> as $|a_n - a_m| \leq \frac{|b-a|}{2^{\min\{n,m\}}}$, so converges to a say.

<u>Claim</u>: a is an accumulation point, as $(a-\epsilon, a+\epsilon)$ contains infinitely many (a_n) terms. $\square$.

§5 Continuous functions

<u>Notation</u> $f: E \subseteq \mathbb{R} \to \mathbb{R}$ means $f: E \to \mathbb{R}$ and $E \subseteq \mathbb{R}$.

§5.1 Introduction to limits

want to define $\lim_{x \to x_0} f(x)$. In calculus, usually require f defined on an open interval $(x_0 - \delta, x_0 + \delta)$, but don't need $x_0 \in \text{domain}(f)$, just "close".

<u>Def^n^</u> (ϵ-δ definition). Suppose $f: E \subseteq \mathbb{R} \to \mathbb{R}$ is a function with domain E, and suppose x_0 is an accumulation point of E. <u>The function f has limit L as x approaches x_0</u>, written $\lim_{x \to x_0} f(x)$, iff, for every $\epsilon > 0$ there is a $\delta > 0$,