Q: if a set is not closed is it open?
open closed?

$[a,b) \leftarrow$ neither open nor closed.

Q: can a set be open and closed? $\phi, \mathbb{R}$.

Defn $E \subset \mathbb{R}$. $\mathrm{int}(E)$ = set of all interior points of E.

Q: A set is closed if all of its points are accumulation points. False: $\{0\}$ closed.

Q: A set is open if it contains all of its interior points. False: always true for any set.

Q: which of the following sets are open/closed/neither

$(-\infty,0) \cup (0,\infty)$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$(0,1) \cup (1,2) \cup \dots \cup (n,n+1) \cup \dots$

$(\frac{1}{2},1) \cup (\frac{1}{4},\frac{1}{2}) \cup (\frac{1}{8},\frac{1}{4}) \cup \dots$

$\{x \mid |x-\pi|<1\}$

$\{x \mid x^2 < 2\}$

$\mathbb{R} \setminus \mathbb{N}$

$\mathbb{R} \setminus \mathbb{Q}$.

Imagine complicated open sets: open intervals, union of open intervals, union of infinitely many open intervals.

Example: order $\mathbb{Q}$ $(q_1, q_2, \dots)$ and consider $U = \bigcup_{i \in \mathbb{N}} (q_i - \frac{1}{2^i}, q_i + \frac{1}{2^i})$

this is an open set containing $\mathbb{Q}$ of length ≤ 2.

Fact: every open set is a union of intervals.

Thm Let $U \subset \mathbb{R}$ be a non-empty open set. Then there is a unique sequence of disjoint open intervals (a_i, b_i) s.t. $G = \bigcup (a_i, b_i)$. $\square$.

§4.4 Elementary topology

Thm A set $E \subset \mathbb{R}$ is ~~closed~~ open iff $\mathbb{R} \setminus E$ is ~~open~~ closed.

Proof $\Rightarrow$ suppose E is open and $F = \mathbb{R} \setminus E$ is not closed. Then there is an accumulation point f of F which does not lie in F, and so lies in E. But as E open, there is an open nbd $\underset{\subset E}{(a,b)}$ s.t. $f \in (a,b) \subset E$, so f can't be an accumulation point of F $\#$.

$\Leftarrow$ suppose $F = \mathbb{R} \setminus E$ is closed but E is not open, so there is a point $e \in E$ s.t. every open nbd ~~E~~ (a,b) with $e \in (a,b)$ contains points of $\mathbb{R} \setminus E$. This implies e is an accumulation point of $F = \mathbb{R} \setminus E$, but $e \notin F$, contradicts F closed. $\square$.

Properties of open sets

1. $\phi, \mathbb{R}$ open
2. A finite intersection of open sets is open
3. Any arbitrary union of open sets is open.
4. The complement of an open set is closed.

Q: what about infinite intersections?

Properties of closed sets

1. $\phi, \mathbb{R}$ closed
2. ~~An arbitra~~ A finite union of closed sets is closed.
3. An arbitrary intersection of closed sets is closed.
4. The complement of a closed set is open.

Q: what about infinite unions?

4.5 Compactness

Q: if a function is locally bounded on $E \subset \mathbb{R}$, is it globally bounded?

Locally bounded: for each $x \in E$, there is an open nbd (a,b) s.t. $x \in (a,b)$ and a number M s.t. $f(x) \leq M$ on (a,b).

A: yes, if $E = [0,1]$ or any closed bounded interval.
no, if $E = (0,1)$ or $\mathbb{R}$.

Examples $f(x) = \frac{1}{x}$ on $(0,1)$ or $f(x) = x$ on $\mathbb{R}$. $\leftarrow$ check locally bounded.

Thm Suppose a function f is locally bounded at each point of a closed and bounded set E. Then f is bounded on E.

§4.5.1 Bolzano–Weierstrass

Thm $E \subset \mathbb{R}$ is closed an bounded iff every sequence in E has a subsequence which converges to E.