Equivalent def$^{\underline{n}}$s

Def$^{\underline{n}}$ β is an eventual ~~limit~~ upper bound of a sequence (x_n) if there is an $N(\beta)$ s.t. $x_n \le \beta$ for all $n \geq N(\beta)$.
$$\limsup_{n\to\infty} x_n = \inf\{\beta \mid \beta \text{ is an eventual upper bound}\}.$$

α is an eventual lower bound for (x_n) if there is an $N(\alpha)$ s.t. $\alpha \le x_n$ for all $n \geq N(\alpha)$.
$$\liminf_{n\to\infty} x_n = \sup\{\alpha \mid \alpha \text{ is an eventual lower bound}\}.$$

Observation: if β is an eventual upper bound so is any larger number, so any number $> \limsup$ is an eventual upper bound

Def$^{\underline{n}}$ A number t is a subsequential limit of (x_n) is there is a subsequence converging to t. Let T be the set of all subsequential limits.
$$\limsup_{n\to\infty} x_n = \sup T \qquad \liminf_{n\to\infty} = \inf T.$$

Th$^{\underline{m}}$ The three definitions of $\limsup / \liminf$ are equivalent.

Th$^{\underline{m}}$ (x_n) converges (to L) iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L \ (\neq \pm\infty)$ ~~$= \lim_{n\to\infty} x_n$~~.

Proof $\Rightarrow$ suppose $\lim_{n\to\infty} x_n = L$. then for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t. $|x_n - L| < \epsilon$ for all $n \geq N$. This implies
$$\sup\{x_N, x_{N+1}, \dots\} \le L+\epsilon \Rightarrow \limsup_{n\to\infty} x_n \le L$$
$$\inf\{x_N, x_{N+1}, \dots\} \geq L-\epsilon. \Rightarrow \liminf_{n\to\infty} x_n \geq L.$$

$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \Rightarrow \limsup x_n = \liminf x_n = L$.

$\Leftarrow$ suppose $\limsup_{n\to\infty} x_n = L$. Then for any $\epsilon > 0$, there is an N_1 s.t.
$\sup\{x_{N_1}, x_{N_1+1}, \dots\} < L+\epsilon$, decreasing $\Rightarrow$ holds for all $n \geq N_1$.

$\liminf x_n = L$. Then for any $\epsilon > 0$ there is an N_2 s.t.
$\inf\{x_{N_2}, x_{N_2+1}, \dots\} \geq L-\epsilon$, increasing $\Rightarrow$ holds for all $n \geq N_2$.

so for all $n \geq \max\{N_1, N_2\}$. $L-\epsilon \le \inf\{x_n, x_{n+1}, \dots\} \le \sup\{x_n, x_{n+1}, \dots\} \le L+\epsilon$

$\Rightarrow L-\epsilon < x_n < L+\epsilon$ for all $n \geq N$, so $|x_n - L| \le \epsilon$ for all $n \geq N$. $\square$

§ 2.12 Cauchy sequences

Motivation: points of a sequence are getting closer together - how do you actually show they converge? To use definition directly need to know L. Monotonic convergence works if (s_n) monotonic. Example $s_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots + \frac{1}{n^2}$

Def^n A sequence (s_n) is called a Cauchy sequence if for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ s.t. $|s_m - s_n| < \epsilon$ for all $m, n \geq N$

Q: · how does this differ from def^n of convergence?

· is it enough to say for all $\epsilon > 0$ $|s_{n+1} - s_n| < \epsilon$?

No: $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ diverges.

Note: indices m, n must be independent.

Examples 1. $s_n = \frac{1}{n}$ is Cauchy: $\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right|$

2. $s_n = 1 + \frac{1}{4} + \dots + \frac{1}{n^2}$ is Cauchy: $\underset{(n>m)}{|s_n - s_m|} = \frac{1}{(m+1)^2} + \dots + \frac{1}{n^2}$

use: $\frac{1}{n^2} \leq \frac{1}{n^2 - n} = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

3. $s_n = \frac{1}{1!} - \frac{1}{2!} + \dots + \frac{(-1)^n}{n!}$ is Cauchy: $2^{r-1} \leq r$ (prove by induction).

4. $s_n = (-1)^n$ not Cauchy.

Th^m Every Cauchy sequence is bounded.

Proof HW 2.12.4. □

Th^m A sequence of real numbers is convergent iff it is Cauchy.

Note: uses completeness of $\mathbb{R}$ - not true in $\mathbb{Q}$.

Proof: $\Rightarrow$ suppose $s_n \to L$. Choose $\frac{\epsilon}{2} > 0$, then there is an $N\left(\frac{\epsilon}{2}\right)$ s.t. $|s_n - L| < \frac{\epsilon}{2}$ for all $n \geq N$. Then $|s_n - s_m| \leq |s_n - L + L - s_m| \leq |s_n - L| + |L - s_m| \leq \epsilon$ for all $n, m \geq N$.