Uses:

1. show divergence by showing a subsequence diverges.
2. show divergence by showing two subsequences converging to different values.
3. if you know (s_n) converges, then any subsequence can be used to find the limit.

Thm (Monotone subsequence thm) Every sequence contains a monotonic subsequence.

Proof we say ~~the~~ s_m is a <u>turn-back point</u> if no later elements are larger, i.e. for all $n \geq m$ $s_m \geq s_n$.

Case 1: there are infinitely many turn-back points $s_{n_1}, s_{n_2}, \ldots$ then this is a non-increasing subsequence as $s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \ldots$.

case 2: finitely many turn back points, let s_M be the last turn back point. ~~then there is $s_{k_1} > s_{M+1}$~~ then s_{M+1} is not a turn back point, so there is $k_1 > M+1$ with $s_{k_1} > s_{M+1}$. s_{k+1} not a turn back point, so there is $k+2 > k+1$ with $s_{k+2} > s_{k+1}$. this constructs an increasing subsequence $s_{k_1} > s_{k_2} > \cdots$ □.

Thm [Bolzano-Weierstrass] Every bounded sequence contains a convergent subsequence.

Proof We have shown every sequence contains a monotonic subsequence. A monotonic sequence which is bounded converges □

HW 2.11. Q 3, 6, 7, 12, 13, 18

§2.13 Upper and lower limits

Defn

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\}$$

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} \inf\{s_n, s_{n+1}, s_{n+2}, \ldots\}$$

Q: Why do these limits exist? $\sup\{s_n, s_{n+1}, \cdots\} \geq \sup\{s_{n+1}, s_{n+2}, \cdots\}$.

§2.13 Upper and lower limits

Defn $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$

Examples

		lim ~~sup~~ inf	lim sup
	$0, 1, 0, 1, \dots$	0	1
$1, \frac{1}{n}$	$1, 1, 1, \frac{1}{2}, 1, \frac{1}{3}, \dots$	0	1
	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$	0	0
	$1, 2, 3, 4, \dots$	$+\infty$	$+\infty$
$-\frac{1}{n}, 1+\frac{1}{n}$		0	1.

Q: why do these limits exist?

set $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$y_{n+1} = \sup\{x_{n+1}, x_{n+2}, \dots\}$ so $y_n \geq y_{n+1}$

so y_n is a ~~decreasing~~ non increasing sequence. If y_n bounded below, then converges, by monotone convergence of bounded sequences. If y_n not bounded below then $y_n \to -\infty$ as $n \to \infty$.

similarly for lim inf.

Thm $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

Proof $\inf\{x_n, x_{n+1}, \dots\} \leq \sup\{x_n, x_{n+1}, \dots\}$. for each n

$\inf\{x_m, x_{m+1}, \dots\} \leq \sup\{x_n, x_{n+1}, \dots\}$. for all n, m !

spse $m<n$: $\quad = \min\{x_m, x_{m+1}, \dots \inf\{x_n, x_{n+1}, \dots\}\}$.

note: $a \leq y_n$ for all y_n, then $a \leq \lim_{n\to\infty} y_n$.

so $\inf\{x_m, x_{m+1}, \dots\} \leq \limsup_{n\to\infty} x_n$.

note: $z_n \leq b$ for all n then $\lim_{n\to\infty} z_n \leq b$

so $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ $\square$.