§2.9 Monotone convergence criterion

Def^n A sequence $(s_n)_{n\in\mathbb{N}}$ is *increasing* if $s_n < s_{n+1}$ for all n

decreasing $s_n > s_{n+1}$ for all n

non-decreasing $s_n \le s_{n+1}$ for all n

non-increasing $s_n \geq s_{n+1}$ for all n

Warning some people say: increasing $\leftrightarrow$ strictly increasing

non-decreasing $\leftrightarrow$ increasing.

Def^n A sequence is *monotonic* if it has any of these four properties.

Def^n A sequence $(s_n)_{n\in\mathbb{N}}$ is *eventually increasing* if there is an N s.t. (s_n) is increasing for all $n \geq N$.

Examples.

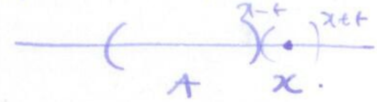
Th^m (*Monotone convergence theorem*)

A monotonic sequence (s_n) is convergent iff it is bounded.

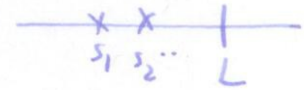
More specifically:

1. If (s_n) is nondecreasing, then there a two possibilities:
   a) (s_n) is bounded above by M, and $s_n \longrightarrow \sup\{s_n\} \le M$.
   b) (s_n) is unbounded and $s_n \longrightarrow +\infty$.
2. same for non-~~decre~~ increasing.

recall: "sup thy" $A \subset \mathbb{R}$, $x = \sup(A)$ iff x is an upper bound, and for all $\epsilon > 0$ there is an $a \in A$ s.t. $a > x - \epsilon$.



<u>Proof</u> · assume (s_n) is bounded, and set $L = \sup\{s_n\}$



for any $\epsilon > 0$ there is an s_N s.t. $s_N > L - \epsilon$.

(s_n) non-decreasing means $s_N \leq s_n$ for all $n \geq N$, so $L - \epsilon < s_N \leq s_n \leq L < L + \epsilon$

so $|s_n - L| < \epsilon$ for all $n \geq N$, as required. $\square$.

· suppose (s_n) is not bounded, so for any $M \in \mathbb{R}$, there is s_N with $s_N \geq M$. (s_n) non-decreasing means $s_n \geq s_N$ for all $n \geq N$

so $s_n \geq s_N \geq M$ for all $n \geq N$, so $(s_n) \to +\infty$. $\square$.

Examples

1. $s_n = \dfrac{n!}{1.3.5\ldots\ldots(2n-1)}$ show decreasing by induction.

2. $s_n = \dfrac{n^2}{2^n}$

3. $x_1 = \sqrt{2}$, $x_2 = \sqrt{2+\sqrt{2}}$, $x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$ $\ldots$ $x_{n+1} = \sqrt{2 + x_n}$.

show increasing by induction: base case $\sqrt{2+\sqrt{2}} > \sqrt{2}$ ✓.

assume $x_k > x_{k-1}$ then $2 + x_k > 2 + x_{k-1}$

so $\sqrt{2+x_k} > \sqrt{2+x_{k-1}}$.

$x_{k+1} > x_k$. ✓.

show bounded above. induction: base: $x_1 = \sqrt{2} < 10$

if $x_n < 10$ then $x_{n+1} = \sqrt{2+x_n} < \sqrt{2+10} = \sqrt{12} < 10$.

this shows x_n converges. (to $L < 10$).

[find L: $x_{n+1} = \sqrt{2+x_n} \Rightarrow (x_{n+1})^2 = 2 + x_n$ take limits. $L^2 = 2 + L$
$L = -1$ or 2].