Limit rules for divergence to $+\infty$

recall difference between divergence (= not convergence) and divergence to $+\infty$. Examples.

Thm If a sequence (a_n) diverges to $+\infty$ and a sequence (b_n) is bounded below by K, then

1. $\{a_n + b_n\}$ diverges to $+\infty$
2. $\{a_n b_n\}$ diverges to $+\infty$ if $K > 0$
3. $\{c a_n\}$ diverges to $+\infty$ for all constants $c > 0$
4. $\{c a_n\}$ diverges to $-\infty$ for all constants $c < 0$.

Proof (of 3.)

(a_n) diverges to $+\infty$ means for all $M > 0$ there is an N s.t. $a_n \geq M$ for all $n \geq N$.

so for all $\frac{M}{c} > 0$ there is an N s.t. $a_n \geq \frac{M}{c}$ for all $n \geq N$

$\Rightarrow$ $c a_n \geq M$ for all $n \geq N$. $\square$. HW 2.7.1, 4, 5, 7

§2.8 Order properties of limits

Convergent comparison Thm Suppose (s_n), (t_n) converge, and $s_n \leq t_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$

Proof pick N s.t. $|s_n - S| \leq \frac{\epsilon}{2}$ for all $n \geq N$
$|t_n - T| \leq \frac{\epsilon}{2}$ for all $n \geq N$

$s_n \leq t_n \Leftrightarrow 0 \leq t_n - s_n$

$$0 \leq T + \underbrace{(t_n - T)}_{\leq \frac{\epsilon}{2}} - S + \underbrace{(S - s_n)}_{\leq \frac{\epsilon}{2}}$$

$$0 \le T - S + \epsilon$$

$$-\epsilon \le T - S \quad \text{for } \underline{\text{all}} \ \epsilon > 0. \Rightarrow \quad 0 \le T - S$$

$$\Rightarrow \quad T \geqslant S. \quad \square.$$

(suppose $T - S < 0$ then false for $\epsilon < |T - S|$).

__Notes__ 1. only need for all n sufficiently large in hypotheses, i.e. for all $n \geqslant N$.

2. __not__ true with strict inequality! $s_n < t_n \not\Rightarrow \lim_{n \to \infty} s_n < \lim_{n \to \infty} t_n$

— find counterexample.

__Corollary__ Suppose (s_n) is convergent, and $\alpha \le s_n \le \beta$ for all n.

Then $\alpha \le \lim_{n \to \infty} s_n \le \beta$.

__Divergent comparison theorem__

__Thm__ If $\{a_n\}$ diverges to $+\infty$, and $a_n \le b_n$ for all n, then (b_n) diverges to $+\infty$.

__Proof__ (sketch) for all $M \in \mathbb{R}$ there is an N s.t. $a_n \geqslant M$. then $b_n \geqslant a_n \geqslant M$. $\square$.

__Example__ show if $r > 1$, then $s_n = r^n$ diverges to $+\infty$.

write $r = 1 + h$, $h > 0$ then

$$r^n = (1+h)^n = 1 + nh + \cdots \geqslant 1 + nh = t_n$$

show (t_n) diverges to $+\infty$, then use comparison ~~that~~ theorem. $\square$.

<u>Squeeze Th^m</u> Suppose $\{a_n\}$, (b_n) and (c_n) are three sequences s.t. $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$. If (a_n) and (c_n) converge to s, then (b_n) must converge to s.

<u>Proof</u> pick $\epsilon > 0$, and choose N_1 s.t. $|a_n - s| < \epsilon$ for all $n \geq N_1$

N_2 s.t. $|b_n - s| \le \epsilon$ for all $n \geq N_2$

choose $N = \max\{N_1, N_2\}$

Then for all $n \geq N$ $\quad s - \epsilon \le a_n \le s + \epsilon$

$s - \epsilon \le c_n \le s + \epsilon$

so $\quad s - \epsilon \le a_n \le b_n \le c_n \le s + \epsilon$

so $\quad s - \epsilon \le b_n \le s + \epsilon \Rightarrow |b_n - s| \le \epsilon$.

so (b_n) converges to s. $\square$.

<u>Example</u> Let θ be a real number. Find $\lim_{n \to \infty} \frac{\sin(n\theta)}{n}$

<u>Absolute value theorem</u> Suppose (s_n) converges. Then $(|s_n|)$ converges and $\lim_{n \to \infty} |s_n| = \left|\lim_{n \to \infty} s_n\right|$

<u>Proof</u> triangle inequality: $|x| + |y| \geq |x + y|$ alternate form: $|a - b| \geq \big||a| - |b|\big|$

(choose $x = a - b$, $y = b$, etc).

for all $\epsilon > 0$ there is an N s.t. $|s_n - s| \le \epsilon$ for all $n \geq N$

$$\big||s_n| - |s|\big| \le |s_n - s| \le \epsilon \quad \text{for all } n \geq N$$

$\Rightarrow \lim_{n \to \infty} |s_n| = |s|$. $\square$.

<u>HW</u> 2.8: 2, 5, 9 and 6, 7, 8 give conclusion but only prove two of these.