§ 2.7 Algebra of limits

(26)

Motivation: • calc I $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2+1/n^2}$ $\lim \frac{1}{n^2} = 0$

$$= \frac{1}{2+\lim 1/n^2} = \frac{1}{2}.$$

• sequence $s_n = \frac{n^2}{2n^2+1}$ obtained from simpler ones by arithmetic operations addition / multiplication / division.

• need to check this works for our definition of a limit.

Thm Suppose (s_n) and (t_n) are convergent sequences, then

1) if $c \in \mathbb{R}$ then $\lim_{n\to\infty} c s_n = c \lim_{n\to\infty} s_n$

2) $\lim_{n\to\infty} (s_n \pm t_n) = \lim_{n\to\infty} s_n \pm \lim_{n\to\infty} t_n$

3) $\lim_{n\to\infty} s_n t_n = \lim_{n\to\infty} s_n \lim_{n\to\infty} t_n$

4) $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$, as long as $\lim_{n\to\infty} t_n \neq 0$

5) $\lim_{n\to\infty} (s_n)^p = \left(\lim_{n\to\infty} s_n\right)^p$ $p \in \mathbb{Z}$

6) $\lim_{n\to\infty} \sqrt[k]{s_n} = \sqrt[k]{\lim_{n\to\infty} s_n}$ if you can take $\sqrt[k]{\cdot}$ roots.

Proofs 1) note: if $|s_n - L| \leq \epsilon$ then $c|s_n - L| \leq \epsilon c$

$$|cs_n - cL| \leq c\epsilon.$$

2) pick $\frac{\epsilon}{2}$ for each lim, then use triangle inequality.

$|s_n - S| \leq \frac{\epsilon}{2}$, $|t_n - T| \leq \frac{\epsilon}{2}$ $\Rightarrow$ $|s_n + t_n - S - T| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

3) First: work out details on scratch paper, then write out proof.

use: $$|s_n t_n - ST| = |s_n t_n - t_n S + t_n S - ST| \quad \leftarrow \text{want this} < \epsilon.$$
$$= |t_n(s_n - S) + S(t_n - T)|$$
$$\le |t_n(s_n - S)| + |S(t_n - T)|.$$
$$\le |t_n||s_n - S| + |S||t_n - T|$$

Q: what do we need to make $|s_n - S|$ and $|t_n - T|$ less than to guarantee above is $< \epsilon$?

$$|t_n||s_n - S| \le \frac{\epsilon}{2} \qquad |S||t_n - T| \le \frac{\epsilon}{2}.$$

$$|s_n - S| \le \frac{\epsilon}{2|t_n|} \le \frac{\epsilon}{2M} \qquad |t_n - T| \le \frac{\epsilon}{2|S|} \le \frac{\epsilon}{2|S|+1}$$

where M is a bound for $|t_n|$.

in case $S = 0$.

4) $$\left|\frac{s_n}{t_n} - \frac{S}{T}\right| = \left|\frac{s_n}{t_n} - \frac{S}{t_n} + \frac{S}{t_n} - \frac{S}{T}\right| \qquad (T \neq 0!).$$

$$= \left|\frac{s_n - S}{t_n} + \frac{S(T - t_n)}{t_n T}\right|$$

$$\le \underbrace{\frac{|s_n - S|}{|t_n|}}_{<\epsilon/2} + \underbrace{\frac{|S|}{|T|}\frac{|t_n - T|}{|t_n|}}_{<\epsilon/2}$$

want:

recall: nonzero limit thm: there is an N s.t. $|t_n| > \frac{|T|}{2}$ for all $n \ge N$

$$\Leftrightarrow \frac{1}{|t_n|} < \frac{2}{|T|}.$$

so $|s_n - S| \le \frac{\epsilon}{2}|t_n| < \frac{\epsilon|T|}{4}$ and $|t_n - T| < \frac{\epsilon}{2}\frac{|T|}{(|S|+1)}|t_n| \le \frac{\epsilon}{2}\frac{|T|}{|S|+1}\frac{|T|}{2}$ $\square$.