Proof $\lim_{n\to\infty} s_n = L_1$ means for any $\epsilon > 0$ $\exists N$ s.t. $|s_n - L_1| < \epsilon$ for all $n \geq N_1(\epsilon)$. Similarly $\lim_{n\to\infty} s_n = L_2$ means for any $\epsilon > 0$ $\exists N$ s.t. $|s_n - L_2| < \epsilon$ for all $n \geq N_2(\epsilon)$

triangle inequality: $|L_1 - L_2| \leq |L_1 - s_n| + |s_n - L_2|$

then for all $n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}$ $|L_1 - L_2| \leq \epsilon + \epsilon \leq 2\epsilon$.

so $|L_1 - L_2| \leq 2\epsilon$ for all $\epsilon > 0$ so $\underset{|L_1-L_2|=0}{\Rightarrow} L_1 = L_2$. □.

Group exercise (if time)

Show that $(s_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ $(2s_n)_{n\in\mathbb{N}}$ convergent.

HW §2.4 2, 3, 5, 8
10, 12, 16.

§2.5 Divergence

Defn If the sequence $(s_n)_{n\in\mathbb{N}}$ does not converge, then we say it diverges.

Quantifiers: convergence for all ($\forall$) $\epsilon > 0$ there is an ($\exists$) N s.t., $|s_n - L| \leq \epsilon$ for all $n \geq N$ ($\forall$),

divergence is not convergence: there exists $\epsilon > 0$ st. for all N there is an $n \geq N$ s.t. $|s_n - L| > \epsilon$.

Examples $s_n = (-1)^n$, $s_n = \sin(n)$, $\underbrace{s_n = n}_{\text{diverges in an "interesting" way}}$

intuition: often want to consider s_n where "$s_n \to \infty$".

Def$^{\underline{n}}$ (Divergence to ∞) Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say (s_n) diverges to ∞, written $\lim_{n\to\infty} s_n = \infty$ or $s_n \to \infty$ as $n \to \infty$.

if for every real number M there is an integer N s.t

$s_n \geq M$ for all $n \geq N$.

Def$^{\underline{n}}$ (divergence to $-\infty$) similar...

Example $s_n = n$ (diverges to ∞) $s_n = -n$ (diverges to $-\infty$)

$s_n = (-1)^n$ does not diverge to $\pm\infty$, just diverges.

Example show $s_n = \dfrac{n^2+1}{n-2}$ diverges to $+\infty$.

recall $\dfrac{a}{b} > \dfrac{c}{d}$ if $a > c$ and $b < d$ $\qquad \dfrac{n^2+1}{n-2} > \dfrac{n^2}{n} = n \geq M$

so can choose $N = M$.

check for all $n \geq N$ $\qquad \dfrac{n^2+1}{n-2} \geq \dfrac{N^2}{N} = N \geq M$. $\square$.

Example $s_n = \dfrac{n^2-1}{n+2}$ find $N(M)$: $\dfrac{n^2-1}{n+2} \geq M$

$n^2 - 1 \geq Mn + 2M$.

$n^2 - Mn \geq 2M + 1$.

$n(n-M) \geq 2M+1$

choose $n \geq 2M+1$

$n - M \geq 1 \Rightarrow n \geq M+1$ $\qquad$ so $N(M) = 2M+1$ works.

check. $\square$.