Notes:

1. N depends on ϵ, if you chose a smaller ϵ you may need a bigger N, so in fact $N(\epsilon) \leftarrow N$ function of ϵ.
2. If some N works for given ϵ, any larger N also works
3. Just need to find some N which works, do not need to find smallest N.

<u>Example</u> $(s_n)_{n\in\mathbb{N}}$ where $s_n = \dfrac{2n+1}{n+2}$ <u>note</u>: $\lim_{n\to\infty} s_n = 2$.

given $\epsilon > 0$ $\exists N$ s.t. $|s_n - 2| \le \epsilon$. for all $n \geq N$.

$(s_n) = \left(1, \frac{5}{4}, \frac{7}{5}, \frac{9}{6}, \dots\right)$

<u>try</u> $\epsilon = \frac{1}{2}$ find N s.t. $\left|\dfrac{2n+1}{n+2} - 2\right| \le \frac{1}{2}$ for all $n \geq N$.

$$\left|\frac{2n+1-2n-4}{n+2}\right| = \left|\frac{-3}{n+2}\right| = \frac{3}{n+2} \le \frac{1}{2}. \quad N = 2 \text{ works.}$$

$\epsilon = \frac{1}{100}$. $\qquad \dfrac{3}{n+2} \le \dfrac{1}{100}$ $\qquad N = 300$ works.

<u>Proof</u> (using Defn).

consider $\left|\dfrac{2n+1}{n+2} - 2\right| = \dfrac{3}{n+2}$ for $n \in \mathbb{N}$.

want $\dfrac{3}{n+2} \le \epsilon$ so $n \geqslant \dfrac{3}{\epsilon} - 2$

$3 \le \epsilon(n+2)$ can chose $N \geqslant \dfrac{3}{\epsilon}$ any ~~largest~~ integer $\left\lceil \dfrac{3}{\epsilon} \right\rceil$.

$3 - 2\epsilon \le \epsilon n$ $\qquad \square$.

$\dfrac{3}{\epsilon} - 2 \le n$

Example $(s_n)_{n\in\mathbb{N}}$ where $s_n = \dfrac{n-4}{2n^2+3}$. $\lim_{n\to\infty} s_n = 0$

Proof find $N(\epsilon)$ s.t. $\left|\dfrac{n-4}{2n^2+3}\right| \leq \epsilon$ for all $n \geq N$.

can choose $N > 4$. s.t. $\dfrac{n-4}{2n^2+3} \leq \epsilon$. first find $N(\epsilon)$:

$$n-4 \leq 2\epsilon n^2 + 3\epsilon$$

$$-4-3\epsilon \leq 2\epsilon n^2 - n = n\underbrace{(2\epsilon n - 1)}_{\text{need this} \geq 0 \text{ or } 1}$$

$2\epsilon n - 1 \geq 0$ $\quad 2\epsilon n > 1$ or 2

$n > \dfrac{1}{2\epsilon}$ or $\dfrac{1}{\epsilon}$.

so $N(\epsilon) = \dfrac{1}{2\epsilon} + 4$ should work:

in fact do $N(\epsilon) = \dfrac{1}{\epsilon}$ and assume $\epsilon < \dfrac{1}{4}$, i.e. $N \geq 4$.

note $\dfrac{a}{b} < \dfrac{c}{d}$ if $a < c$ and $b > d$.

$$\frac{n-4}{2n^2+3} < \frac{1-\frac{4}{n}}{2n+\,^{3}/_{n}} < \frac{1}{^{2}/_{\epsilon}} = \frac{\epsilon}{2} < \epsilon \quad \text{as required } \square.$$

Example $s_n = \dfrac{1}{n} - \dfrac{1}{n+1}$ use: $\left|\dfrac{1}{n} - \dfrac{1}{n+1}\right| \leq \left|\dfrac{1}{n}\right| + \left|\dfrac{1}{n+1}\right|$

$\leq \dfrac{1}{2n} < \epsilon$ choose $N(\epsilon) = \dfrac{1}{2\epsilon}$ $\square$.

Thm (Uniqueness of limits) If

$\lim_{n\to\infty} s_n = L_1$ and $\lim_{n\to\infty} s_n = L_2$ then $L_1 = L_2$.