

$$\sqrt{2} = 1.41421356237 \dots$$

(17)

get 3 decimal places of accuracy in 4 steps.

4

5

47

≥ 7 steps. $\leftarrow$ this suffices for any application.

Main Q: does this go on for ever?

i.e. do we always get increasing accuracy

i.e. for an error tolerance $\epsilon > 0$, is there an N s.t. $|x_n - \sqrt{2}| \leq \epsilon$ for all $n \geq N$?

If so, then this sequence $(x_n)_{n \in \mathbb{N}}$ converges to $\sqrt{2}$, and really determines $\sqrt{2}$, and not any other number.

§ 2.2 Sequences A sequence of numbers $(x_n)_{n \in \mathbb{N}}$ is ^{advised} a list of numbers.

Defn A sequence of real numbers is a function $f: \mathbb{N} \rightarrow \mathbb{R}$.

Notation: usually write f_i instead of $f(i)$. etc.

Warning: people often write $\{x_n\}$ for the sequence (x_n) .

Note: $\{0, 0, 0, \dots\}$ doesn't make sense as a set ($= \{0\}$) but is a perfectly good sequence $(x_n)_{n \in \mathbb{N}}$, $x_n = 0$.

Examples

Informal notation: $2, 4, 6, 8, 10, \dots$

consider the sequence of ^{positive} even integers.

more formal

$$(x_n), \quad x_n = 2n.$$

equivalent

$$x_1 = 2, \quad x_{n+1} = x_n + 2 \quad (\text{implicit / recursive})$$

arithmetic progressions: $x_n = c + nd$

geometric progressions: $x_n = ar^n$

iterated functions: $x_1 = c, x_{n+1} = f(x_n)$

(i.e.) $c, f(c), f(f(c)), f(f(f(c))), \dots$

sequences of partial sums:

if (x_n) is a sequence, then the sequence of partial sums is (s_n)

$$s_1 = x_1 \quad s_{n+1} = s_n + x_{n+1}$$

$$s_2 = x_1 + x_2$$

$$s_3 = x_1 + x_2 + x_3, \dots \in \mathbb{R}.$$

HW. 2.2.6

Remark $x_n = \frac{1}{n}$ $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ } no simple formula...

§ 2.3 Countable sets.

Defn An infinite set A is countable if there is a bijection $f: \mathbb{N} \rightarrow A$.

Examples. $\mathbb{N}, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}$.

Fact the set $\mathbb{R}$ of real numbers is not countable. [Cantor].

Proof suffices to show this for $(0,1) \subset \mathbb{R}$.

Suppose there is a sequence (s_n) which contains every element in $(0,1)$

$$s_1 = 0.x_{11}x_{12}x_{13}\dots$$

$$\text{consider } c = 0.c_1c_2c_3c_4\dots$$

$$s_2 = 0.x_{21}x_{22}x_{23}\dots$$

$$\text{where } c_i = \begin{cases} 0 & \text{if } x_{ii} \neq 0 \\ 1 & \text{if } x_{ii} = 0 \end{cases}$$

$$s_3 = 0.x_{31}x_{32}x_{33}\dots$$

then $c \notin (s_n)$ $\square$.

HW 2.3.10

§ 2.4 Convergence

Examples $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \dots$ gets closer and closer to 0.

we say $(x_n) = (\frac{1}{n})$ converges to 0.

intuition: " (x_n) converges to L if the numbers x_n get closer and closer to L "

Example $0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, \dots$

- get close to 0, but not steadily close to 0.

$0.1, 0.11, 0.111, 0.1111, \dots$

- gets close to 2, but not very close. gets close to $\frac{1}{9} = 0.11111\dots$!

Defn (Cauchy) A sequence $(s_n)_{n \in \mathbb{N}}$ converges to a real number L

if and only if for every $\epsilon > 0$ there is a positive integer N such that

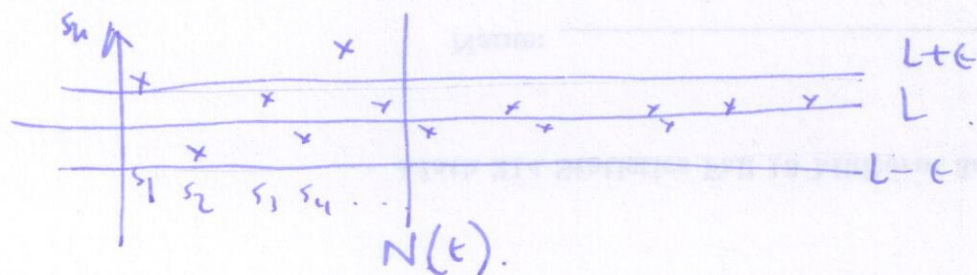
$$|s_n - L| \leq \epsilon \text{ for all } n \geq N.$$

Notation: write $\lim_{n \rightarrow \infty} s_n = L$ or $s_n \rightarrow L$ as $n \rightarrow \infty$.

a sequence which converges is called convergent

"does not converge" divergent

intuition: ϵ is the "error" or "tolerance".



Remark

N depends on ϵ ,
sometimes write
 $N(\epsilon)$ to make
this explicit.