$\sqrt{2} = 1.41421356237\ldots$

get 3 decimal places of accuracy in 4 steps.

4   5

47   ≥ 7 steps. ← this suffices for any application.

<u>Main Q</u>: does this go on for ever?

i.e. do we always get increasing accuracy

i.e. for an error tolerance $\epsilon > 0$, is there an N s.t. $|x_n - \sqrt{2}| \leq \epsilon$ for <u>all</u> $n \geq N$?

If so, then this sequence $(x_n)_{n\in\mathbb{N}}$ <u>converges</u> to $\sqrt{2}$, and really determines $\sqrt{2}$, and not any other number.

<u>§2.2 Sequences</u> A sequence of numbers $(x_n)_{n\in\mathbb{N}}$ is an endless list of numbers.

<u>Defn</u> A <u>sequence</u> of real numbers is a function $f: \mathbb{N} \to \mathbb{R}$.

<u>Notation</u>: usually write f_1 instead of $f(1)$. etc.

<u>Warning</u>: people often write $\{x_n\}$ for the sequence (x_n).

Note: $\{0,0,0 \ldots\}$ doesn't make sense as a set $(=\{0\})$ but is a perfectly good sequence ~~this~~ $(x_n)_{n\in\mathbb{N}}$, $x_n = 0$.

<u>Examples</u> informal notation: $2, 4, 6, 8, 10, \ldots$

consider the sequence of positive even integers.

more formal (x_n), $x_n = 2n$.

equivalent $x_1 = 2$, $x_{n+1} = x_n + 2$ (implicit / recursive) arithmetic progressions : $x_n = c + nd$

geometric progressions : $x_n = ar^n$

iterated functions : $x_1 = c, \quad x_{n+1} = f(x_n)$

$$\text{(i.e.)} \quad c, f(c), f(f(c)), f(f(f(c))), \ldots$$

sequences of partial sums :

if (x_n) is a sequence, then the sequence (s_n) of partial sums is

$s_1 = x_1 \qquad s_{n+1} = s_n + x_{n+1}$

so $\quad s_2 = x_1 + x_2$

$\quad s_3 = x_1 + x_2 + x_3, \ldots$ etc.

HW. 2.2.6

<u>Remark</u> $x_n = \frac{1}{n} \qquad s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ } no simpler formula...

§ 2.3 Countable sets.

<u>Defn</u> An infinite set A is <u>countable</u> if there is a bijection $f : \mathbb{N} \to A$.

<u>Examples</u>. $\mathbb{N}$, $\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{Q}$.

<u>Thm</u> the set $\mathbb{R}$ of real numbers is <u>not</u> countable. [Cantor].

<u>Proof</u> suffices to show this for $(0,1) \subset \mathbb{R}$

Suppose there is a sequence (s_n) which contains every element in $(0,1)$

$s_1 = 0 . x_{11} x_{12} x_{13} \ldots \qquad$ consider $c = 0 . c_1 c_2 c_3 c_4 \ldots$

$s_2 = 0 . x_{21} x_{22} x_{23} \ldots \qquad$ where $c_i = \begin{cases} 0 & \text{if } x_{ii} \neq 0 \\ 1 & \text{if } x_{ii} = 0 \end{cases}$

$s_3 = 0 . x_{31} x_{32} x_{33} \ldots$

then $c \notin (s_n)$ ※ $\square$.

HW 2.3.10

§2.4 Convergence

Examples $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ gets closer and closer to 0.

we say $(x_n) = (\frac{1}{n})$ converges to 0.

intuition: "(x_n) converges to L if the numbers x_n get closer and closer to L"

Example · 0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, ...
– get closer to 0, but not steadily closer to 0.

· 0.1, 0.11, 0.111, 0.1111, ...
– gets closer ~~and~~ to 2, but not very close. gets closer to $\frac{1}{9} = 0.11111\ldots$!

Def$^{\underline{n}}$ (Cauchy) A sequence $(s_n)_{n \in \mathbb{N}}$ converges to a real number L if and only if for every $\epsilon > 0$ there is a positive integer N such that

$$|s_n - L| \leq \epsilon \text{ for all } n \geq N.$$

Notation: write $\lim_{n \to \infty} s_n = L$ or $s_n \to L$ as $n \to \infty$.

Remark N depends on ϵ, sometimes write $N(\epsilon)$ to make this explicit.

a sequence which converges is called convergent

" does not converge " divergent

intuition: ϵ is the "error" or "tolerance".