Note. • $\emptyset$ is bounded; any number is upper bound (and lower bound).

• a given set ~~has~~ may have many upper bounds. e.g. $\{0,1,2\}$ has upper bound 100, 200, 2.

special upper bounds:

Def^{n} If there is a number $M \in E$ s.t. $x \le M$ for all $x \in E$, then we say M is the <u>maximum</u> of E, <u>notation</u> $M = \max E$.

Def^{n} If there is a number $m \in E$ s.t. $m \le x$ for all $x \in E$, then we say m is the <u>minimum</u> of E, written $m = \min E$.

<u>Example</u> $E = [0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ $\max E = 1$, $\min E = 0$

<u>Q</u>: does every set have a max or min?

<u>A</u>: <u>No!</u>

<u>Example</u> $E = (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ $\max E$ DNE, $\min E$ DNE

§1.6 Sups and Infs

<u>Motivation</u>: a set may be bounded above, but with no maximum, and has many upper bounds. Maybe there is a smallest upper bound or least upper bound.

<u>Def^{n}</u> Let E be a set of real numbers, which is bounded above and non-empty. If M is the least of all the upper bounds, then

$M =$ <u>least upper bound</u> of E = <u>supremum</u> of E = $\sup E$

<u>Def^{n}</u> Let E be a set of real numbers, bounded below and non-empty. If ~~M~~ m is the greatest lower bound, then

$M =$ <u>greatest upper bound of E</u> = <u>infimum of E</u> = $\inf E$

we extend the definition to ϕ and unbounded sets by

1. $\inf \phi = \infty$ and $\sup \phi = -\infty$
2. E unbounded above, $\sup E = \infty$
3. E unbounded below, $\inf E = -\infty$.

not real numbers.

Completeness

<u>Completeness axiom</u> A non-empty set of real numbers which is bounded above has a least upper bound. (ie. if $E \neq \phi$ and bounded above then $\sup E$ exists and is a real number).

<u>Important</u>: these are the <u>only</u> properties of the real numbers we may ~~as~~ use!

~~Notes~~ <u>Remarks</u> · can't prove completeness from field axioms.

(in fact $\mathbb{Q}$ is an ordered field where completeness fails, e.g. consider $(0, \sqrt{2})$)

· intuition: completeness means "no holes in the number line".

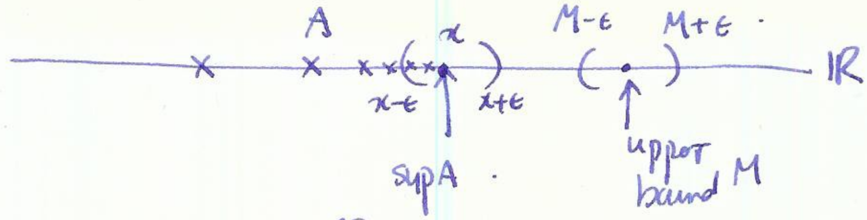
<u>Fact</u> · $\mathbb{R}$ is a complete ordered field

· this essentially characterizes $\mathbb{R}$: $\mathbb{R}$ is the only complete ordered field.

<u>Example</u> (of proving something)

"Sup Thm" Show that if A is a non-empty set of real numbers, a number x is the supremum of A <u>iff</u> $\Leftrightarrow$! x is an upper bound for A and for every $\epsilon > 0$ there is an $a \in A$ s.t. $x - \epsilon < a$

intuition: draw picture



two directions: P Q.

$x = \sup A \iff x$ is an upper bound and for all $\epsilon > 0$ there is $a \in A$ with $x - \epsilon < a$.

contrapositive: $P \Rightarrow Q$ equivalent to not $Q \Rightarrow$ not P

picture: 

(think sets: $A \subset B \iff B^c \subset A^c$

$\Rightarrow$ not/negative statement: x is an upper bound for A and there is an $\epsilon > 0$ such that for all $a \in A$, $x - \epsilon > a$, equivalently $a < x - \epsilon$

but this means $x - \epsilon$ is an upper bound, so x is not the least upper bound, so x is not $\sup A$. (so not $Q \Rightarrow$ not P)

(so we have shown $P \Rightarrow Q$).

$\Leftarrow$ suppose x is not the supremum, then there is a smaller lower bound y say 

. set $\epsilon = \frac{x-y}{2}$, then there is an $a \in A$ with

$x - \epsilon < a$, i.e. $x - \frac{x-y}{2} < a$

$$\frac{x}{2} + \frac{y}{2} < a \quad \text{but} \quad y < x$$

$$\underbrace{\frac{y}{2} + \frac{y}{2}}_{y} < \frac{x}{2} + \frac{y}{2} < a$$

, so $y < a$, contradicts y upper bound. $\square$.

§1.7 Archimedean property

Thm The set $\mathbb{N}$ has no upper bound.

Fact: this looks like it should only depend on algebraic and order structure on $\mathbb{R}$, in fact the proof uses completeness, in an essential way.

(I assume this is because there are ordered fields in which $\mathbb{N}$ has an upper bound – don't know how to construct one offhand...).

<u>Proof</u> Suppose $\mathbb{N}$ has an upper bound, then it has a least upper bound

Let $x = \sup \mathbb{N}$ (a real number).

Then $n \le x$ for all natural numbers $n \in \mathbb{N}$.

but $n \le x-1$ cannot be true for all natural numbers x (recall "sup thing"!)

Choose some natural number m with $m > x-1$

but then $m+1$ is a natural number with $m+1 > x$ # x sup/upper bound. □.

<u>Consequences</u>:

1. No matter how large a real number x is, there is always a natural number n which is larger.
2. Given any positive number y, no matter how large, and any positive number x, no matter how small, can add x to itself sufficiently many times such that $nx > y$.
3. Given any positive number x, no matter how small, there is a fraction $1/n$, n natural number with $\frac{1}{n} < x$.

~~§1.10~~ <u>§1.9 The rational numbers are dense in $\mathbb{R}$</u>

<u>Defn</u> A set E of real numbers is said to be <u>dense</u> in $\mathbb{R}$ if every open interval (a,b) contains a point of E.

recall: $(a \neq b)$

<u>Thm</u> The set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.

<u>Proof</u> Consider the open interval (a,b), ~~with~~ $(\Rightarrow a<b)$.

By the Archimedean property, there is a natural number n s.t.

$$n > \frac{1}{x-y}$$