cases) do calculations with infinite precision.

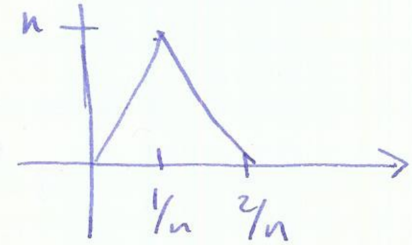
Vagne arguments often wrong

Example Let $f_n : [0,1] \to \mathbb{R}$ be continuous functions for $n \in \mathbb{N}$ and $f : [0,1] \to \mathbb{R}$ another continuous function. If $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in [0,1]$, then the functions f_n are getting close to ~~converging~~ to f so the maxima

$$\left.\begin{array}{l} \max f_n \to \max f \\ \text{and} \quad \int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx \end{array}\right\} \text{wrong!}$$

e.g. consider

$$f_n(x) = \begin{cases} n^2 x & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$



$$f(x) = 0$$

These functions are continuous and $f_n(x) \to f(x)$ for all $x \in [0,1]$ as $n \to \infty$. However

$$\max f_n = n \qquad \max f = 0$$

$$\int_0^1 f_n(x)\,dx = 1 \qquad \int_0^1 f(x)\,dx = 0$$

Problem: what does "getting close to" mean here for functions?

need to replace plausible assertions with precise arguments.

Quantifiers.

we will often say "For all .... it is true that .... "

"There is a .... such that ....

Examples
- For all real numbers $x \in \mathbb{R}$, $(x+1)^2 = x^2 + 2x + 1$
- there is a real number $x \in \mathbb{R}$ such that $x^2 + 2x + 1 = 0$

Notation for all $\leftrightarrow$ $\forall$

there exists $\leftrightarrow$ $\exists$

Examples above are: $\forall x \in \mathbb{R}, \quad (x+1)^2 = x^2 + 2x + 1$

$\exists x \in \mathbb{R}, \quad x^2 + 2x + 1 = 0$

Negation of quantifiers

"all birds fly" has negation "at least one bird does not fly"

↑ complementary statement: at least one of A or not A is true.

More formally:

"For all birds b, b flies" has negation "There exists a bird b, b does not fly".

Notation:

$\forall b$, b flies $\qquad$ $\exists b$, b does not fly.

Note: $\forall b \in B$, "statement about b" is true

has negation: $\exists b \in B$, "statement about b" is false.

So when we take negations, replace $\forall$ by $\exists$ and $\exists$ by $\forall$.

eg $\exists a \in A, \forall b \in B, \forall c \in C$, "statement about a, b, c" is true

has negation $\forall a \in A, \exists b \in B, \exists c \in C$, "statement about a, b, c" is false.

§1 Properties of the real numbers

(tell class to read sections §1.1, 1.2)

We will now clearly state what properties of the real numbers we will use.

§1.3 Algebraic structure

$\mathbb{R}$ is a field (so are number ~~vector~~ fields, vector spaces ~~number fields~~ are not fields but are similar.) binary two operations + addition $\times$ or $\cdot$ multiplication.

Def^{n} The field axioms are

A1: for any $a, b \in \mathbb{R}$ there is a number $a+b \in \mathbb{R}$, and $a+b = b+a$

A2: for any $a, b, c \in \mathbb{R}$ $(a+b)+c = a+(b+c)$

A3: there is a unique number 0 s.t. for all

A4: for any number $a \in \mathbb{R}$ there is a corresponding number $-a$ such that $a + (-a) = 0$

M1: For any $a, b \in \mathbb{R}$ there is a number $ab \in \mathbb{R}$ and $ab = ba$

M2: for any $a, b, c \in \mathbb{R}$, $(ab)c = a(bc)$

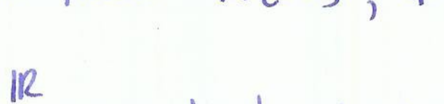
M3: there is a unique number $1 \in \mathbb{R}$ such that $a1 = 1a = a$ for all $a \in \mathbb{R}$.

M4: for any a, ~~b~~ $\in \mathbb{R}$, $a \neq 0$, there is a corresponding number a^{-1} such that $aa^{-1} = 1$

AM1: For any $a, b, c \in \mathbb{R}$, $(a+b)c = ac + bc$.

[other fields: number fields $\mathbb{Q}[\sqrt{2}]$, $\mathbb{C}$, $\mathbb{Q}$, finite fields $\mathbb{Z}/p\mathbb{Z}$]

intuition is always $\mathbb{R}$



, but when we prove things we use the properties...

§1.4 Order structure

Some numbers are "bigger" than others. Here "bigger" means "to the right" on the number line. Notation $x<y$ $(x \neq y)$, $x \leq y$

Defn An ordered field is a field which satisfies the following additional properties.

O1 : For any $a,b \in \mathbb{R}$, exactly one of the following holds:
$a=b$, $a<b$, $b<a$

O2 : For any $a,b,c \in \mathbb{R}$ if $a<b$ and $b<c$ then $a<c$

O3 : For any $a,b,c \in \mathbb{R}$ if $a<b$ then $a+c<b+c$

O4 : For any $a,b \in \mathbb{R}$ with $a<b$, for any $c \in \mathbb{R}$ with $c>0$ then $ac<bc$.

Fact the real numbers $\mathbb{R}$ form an ordered field.

§1.5 Bounds

Let E be a set of real numbers (i.e. $E \subset \mathbb{R}$)

Defn A number M is an upper bound (or bounded above) for E if $x \leq M$ for all $x \in E$

Defn A number m is a lower bound (or bounded below) for E if $m \leq x$ for all $x \in E$

Defn A set E is bounded if it has both upper and lower bounds.

Example $E = \{0,1,2\}$ has upper bound 100, lower bound -100 so is bounded.

$E = (2,\infty)$ has lower bound 0, no upper bound.

$E = \mathbb{N}$ no upper or lower bound.