MTH 341 Advanced Calculus

Joseph Maher joseph.maher@csi.cuny.edu

web page : http://www.math.csi.edu/~maher

office 1S-222 office hours M 2:30-4:25
W 3:35-4:25

– students with disabilities

Text: Elementary Real Analysis, Thomson, Bruckner, Bruckner (available online)

§ Appendix: Background (tell them to read the remaining sections of this)

sets: collections of objects, e.g. numbers. $\{0,1\}$ ← set with two elements, 0 and 1.
or e.g. {banana, apple}.

notation : $x \in A$ x is a number/object in set A

$x \notin A$ x is not in A

examples : $0 \in \{0,1\}$ true.

banana $\notin$ {banana, apple} false.

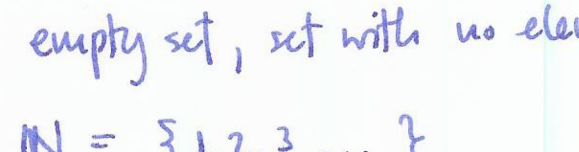
Special sets: · $\emptyset$ empty set, set with no elements, also $\{\}$

· Natural numbers $\mathbb{N} = \{1,2,3,\dots\}$

· Integers $\mathbb{Z} = \{\dots,-1,0,1,2,3,\dots\}$

· Rationals $\mathbb{Q}$ fractions $\frac{m}{n}$, $m,n \in \mathbb{Z}$, $n \neq 0$.

· Real numbers $\mathbb{R}$



· Closed intervals $[a,b]$ = all real numbers $a \leq x \leq b$ (includes end points)

· open intervals (a,b) = all real numbers $a < x < b$ (does not include endpoints)

- infinite intervals (a,∞) = all real numbers $x > a$

  $[a,\infty)$ = all real numbers $x \geq a$

  $(-\infty,a)$ = all real numbers $x < a$

  $(-\infty,a]$ " " $x \leq a$

- sets as lists: $\{1, 4, -2, \sqrt{47}\}$ order doesn't matter.

  $\{1,2\} = \{2,1\}$

- sets defined by descriptions: $\{x \mid |x| < 1\}$ $(= (-1,1))$

  alternate notation: $\{x : |x| < 1\}$

more generally, if $C(x)$ is some statement/property ~~of~~ then

$\{x \mid C(x)\}$ means "all x for which $C(x)$ is true".

Examples

$\{x \mid x^2 - 1 \leq 0\}$ $(= [-1,1])$

$\{x \mid x^2 + 1 \leq 0\}$ $(= \phi$ or $\{\})$.

$\{x \mid x \text{ is a subset of } \{0,1\}\}$ $(= \{\phi, \{0\}, \{1\}, \{0,1\}\})$

Warning ~~&~~ trying to define sets by $\{x \mid C(x)\}$ essentially doesn't work, without some restriction on what $C(x)$ is allowed to be.

Key example (Russell's paradox)

let S_1 be set of all sets: $S_1 = \{S \mid S \text{ is a set}\}$

note: $S_1 \in S_1$

consider $S_2 = \{ s \mid s \text{ is a set, and } s \notin s \}$

Q is $S_2 \in S_2$? ($\notin$)

$$\left.\begin{array}{l} S_2 \in S_2 \Rightarrow S_2 \notin S_2 \\ S_2 \notin S_2 \Rightarrow S_2 \in S_2 \end{array}\right\} \text{ \# contradiction.}$$

Solution: there is ~~set~~ no set of all sets.

define sets carefully, using definite rules (ZFC)

long story — which we shall ignore and just carry on.

Weird fact: this paradox arises in the English language.

autological: an adjective which describes itself (e.g. polysyllabic)

heterological: an adjective which does not describe itself (monosyllabic)

Conclusion: language is not logically consistent.

Set notation

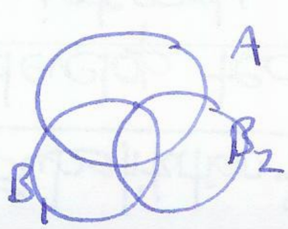
subset	$A \subset B$	every element of A is in B
intersection	$A \cap B$	the set consisting of elements which lie in both A and B.
union	$A \cup B$	the set consisting of elements which lie in either A or B.
differences/complements	$A \setminus B$	the set of elements of A which do not lie in B.

Examples

$$A \setminus (B_1 \cup B_2) = (A \setminus B_1) \cap (A \setminus B_2)$$

$$A \setminus (B_1 \cap B_2) = (A \setminus B_1) \cup (A \setminus B_2)$$

Proof: draw a picture



$\square$

Ordered pairs (a,b) $a \in A$, $b \in B$ order matters (not a set) $(a,b) \neq (b,a)$ in general.

the set of all ordered pairs with $a \in A$ and $b \in B$ is $A \times B$ the Cartesian product of A and B.

Functions ~~$f: A \to B$~~

A function f (or map) from a set A to a set B is a rule (*) that assigns a value $f(a) \in B$ to each element $a \in A$.

notation $\underset{\text{domain}}{f: A} \longrightarrow \underset{\text{range}}{B}$ note f name of function

$f(x)$ value of function at x

(*) what does this mean?

Defn A, B non-empty sets. A set f of ordered pairs (a,b) with $a \in A$ and $b \in B$ is called a function $f: A \to B$ if for every $a \in A$ there is precisely one pair (a,b) in f.

Notation we will still write $f(x) = x^2 + x + 1$ to mean

$$f: \mathbb{R} \to \mathbb{R}, \text{ with } f = \{ (x, x^2+x+1) \mid x \in \mathbb{R} \}.$$

Warning: • domain and range often defined implicitly (i.e. you have to work them out).

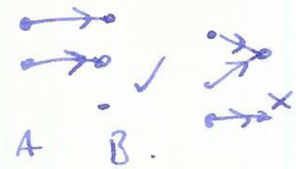
i.e. people write $f(x) = \sqrt{x}$ instead of $f: [0, \infty) \to \mathbb{R}$

• domain vs image. image is all values which arise from the function, i.e. if $f: A \to B$ image$(f) = \{ b \in B \mid b = f(a) \text{ for some } a \in A \}$

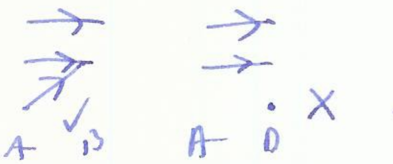
<u>note</u>: $f:[0,\infty)\to\mathbb{R}$, $x\mapsto\sqrt{x}$ and $f:[0,\infty)\to[0,\infty)$, $x\mapsto\sqrt{x}$

are technically different functions, but they have the same image.

A function f is <u>one-to-one</u> (<u>injective</u>, monomorphic) if for each $b\in \text{image}(f)$ there is precisely one $a\in A$ with $f(a)=b$.



A function f is <u>onto</u> (surjective, epimorphism) if for each $b\in B$, there is an $a\in A$ with $f(a)=b$.



<u>Inverses</u> if $f:A\to B$ is one-to-one, then there is an inverse function,

$f^{-1}:\text{image}(A)\to B$ such that $f^{-1}(f(a))=a$ for all $a\in A$.

B

<u>Characteristic functions</u> $A\subset B$ the $\chi_A : B\to\{0,1\}\subset\mathbb{R}$

$$\chi_A(b)=\begin{cases}1 & \text{if } b\in A\\ 0 & \text{if } b\notin A.\end{cases}$$

<u>Composition</u> $f:A\to B$ and $g:B\to C$

then $(g\circ f)(x)=g(f(x))$ so $g\circ f: A\to C$

<u>Real analysis</u> rigorous study of functions $f:\mathbb{R}\to\mathbb{R}$, sequences, series, limits, continuity, etc. We will <u>prove</u> things.

<u>Why proof?</u> Start with careful and precise definitions, and use agreed rules to make deductions that are definitely correct, can (in some