Math 341 Advanced Calculus Spring 13 Midterm 2

Name: ______________________________

- Do any 7 of the following 9 questions.
- You may use a single letter size page of notes, you may write on both sides. You do not need a calculator.

(1) Give the definition of a Cauchy sequence.

(2) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{1}{n^2+1}$ converges.

(3) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{n}{3n+4}$ converges.

(4) Use the definition of convergence to show that if $a_n \to L$ then $|a_n| \to |L|$.

(5) Let a_n be a bounded sequence, and suppose $b_n \to 0$ as $n \to \infty$. Show that $a_n b_n \to 0$.

(6) Let $a_n = (-1)^n + \frac{1}{n}$. Find $\limsup(a_n)$ and $\liminf(a_n)$.

(7) Suppose $\limsup(a_n) = L$. Show that there is a subsequence of (a_n) which converges to L.

(8) Consider the sequence given by $a_1 = 1$ and $a_{n+1} = \frac{1}{4}(a_n + 1)$. Show this sequence is convergent, for example by using monotone convergence, or by any other method.

(9) Let E be a subset of $\mathbb{R}$. Show that a boundary point of E cannot be an interior point of E.

Midterm 2	
Overall	

Midterm 2 Solutions

Q1 (s_n) is a Cauchy sequence if for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t. for all $n, m \geq N$, $|s_n - s_m| \leq \epsilon$.

Q2 Show $s_n = \frac{1}{n^2+1}$ converges to 0, i.e. for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t. for all $n \geq N$, $\frac{1}{n^2+1} < \epsilon$. **Note**: $\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}$ for $n \geq 1$, so $\frac{1}{n} \leq \epsilon \Leftrightarrow n \geq \frac{1}{\epsilon}$.
Given $\epsilon > 0$,
Choose $N = \frac{1}{\epsilon}$, then for all $n \geq N$, $\frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} = \epsilon$, as required.

Q3 Show $s_n = \frac{n}{3n+4}$ converges to $\frac{1}{3}$. **Note**: $\left|\frac{n}{3n+4} - \frac{1}{3}\right| = \left|\frac{3n - 3n - 4}{3(3n+4)}\right| = \left|\frac{4}{9n+12}\right|$

$< \frac{4}{9n} < \epsilon \Rightarrow n > \frac{4}{9\epsilon}$.

Given $\epsilon > 0$, choose $N = \frac{4}{9\epsilon}$, then $|s_n - L| = \left|\frac{n}{3n+4} - \frac{1}{3}\right| = \left|\frac{4}{9n+12}\right| < \frac{4}{9n} \leq \frac{4}{9N} = \epsilon$, as required.

Q4 $a_n \to L$ means for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t. for all $n \geq N$, $|a_n - L| < \epsilon$.
Triangle inequality: $\big||a_n| - |L|\big| \leq |a_n - L|$
so for all $\epsilon > 0$ there is a $N(\epsilon)$ s.t. $\big||a_n| - |L|\big| \leq |a_n - L| < \epsilon$, so $|a_n| \to |L|$, as required.

Q5 $|a_n| \leq M$ for all n. $b_n \to 0$ means for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t. for all $n \geq N$, $|b_n| \leq \epsilon$. **Note**: $|a_n b_n| = |a_n||b_n|$.
Choose $\frac{\epsilon}{M} > 0$, then for all $n \geq N(\epsilon/M)$, $|a_n b_n| \leq |a_n||b_n| \leq M|b_n| \leq M \cdot \epsilon/M = \epsilon$,
so $a_n b_n \to 0$, as required.

Q6 $a_n = (-1)^n + \frac{1}{n}$. $s_n = \sup\left\{(-1)^n + \frac{1}{n}, (-1)^{n+1} + \frac{1}{n+1}, \dots\right\} = \begin{cases} 1 + \frac{1}{n+1} & n \text{ odd} \\ 1 + \frac{1}{n} & n \text{ even} \end{cases} = \left(1+\frac{1}{2}, 1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{4}, \dots\right)$

so $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 + \frac{1}{2n} = 1$.

$i_n = \inf\left\{(-1)^n + \frac{1}{n}, (-1)^{n+1} + \frac{1}{n+1}, \dots\right\} = \inf\left\{-1 + \frac{1}{2n+1}, -1 + \frac{1}{2n+3}, \dots\right\}$ (can ignore +ve terms, $2n+1$ = smallest odd number $\geq n$).

$= -1$, so $\lim_{n\to\infty} i_n = \lim_{n\to\infty} -1 = -1$.

Q7 $\limsup(a_n) = L$ means $\lim_{n\to\infty} s_n = L$ where $s_n = \sup\{a_n, a_{n+1}, \dots\}$

so for all $\epsilon > 0$ there is an N s.t. $|s_n - L| < \epsilon$ for all $n \geq N$.

also for all $\epsilon > 0$, there is an $m > n$ s.t. $a_m > s_n - \epsilon$.

Choose $\epsilon_n = \frac{1}{n}$. For $n = 1$, choose $N_1 = N(\epsilon_1)$, and then ~~choose~~ there is $m_1 > N_1$ s.t. $a_{m_1} > s_{N_1} - \epsilon_1$. For $k \geq 2$, choose $N_k = \max\{N(\epsilon_k), N_{k-1}\}$, and there is an $m_k > N_k$ s.t. $a_{m_k} > s_{N_k} - \epsilon_k$. For the sequence (a_{m_k}), $|a_{m_k} - L| \leq |a_{m_k} - s_{m_k}| + |s_{m_k} - L| \leq \epsilon_k + \epsilon_k = 2\epsilon_k$ for all k, and $\epsilon_k = \frac{1}{k} \to 0$, so $a_{m_k} \to L$.

Q8 $a_1 = 1$ $a_{n+1} = \frac{1}{4}(a_n + 1)$ Claim: (a_n) is decreasing. Induction:

base case: $a_1 = 1$ $a_2 = \frac{1}{2}$, $a_1 > a_2$. induction step: suppose $a_{n+1} < a_n$, then $a_{n+1} + 1 < a_n + 1 \Rightarrow \frac{1}{4}(a_{n+1} + 1) < \frac{1}{4}(a_n + 1) \Rightarrow a_{n+2} < a_{n+1}$, as required.

So (a_n) is decreasing and all terms positive, so bounded below by 0, so (a_n) converges by monotonic convergence.

Q9 $E \subset \mathbb{R}$. $x \in E$ interior point $\Rightarrow$ there is a $c > 0$ s.t. $(x-c, x+c) \subset E$.

$x \in \mathbb{R}$ boundary point if for all $c > 0$, $(x-c, x+c)$ contains points of E, points of $\mathbb{R}\setminus E$.

but if $(x-c, x+c) \subset E$, then $(x-c, x+c)$ contains no points of $\mathbb{R}\setminus E$, so x can't be an boundary point.