Math 341 Advanced Calculus Spring 13 Midterm 1

Name: Solutions

- Do any 7 of the following 9 questions.
- You may use a single letter size page of notes, you may write on both sides. You do not need a calculator.

(1) State the completeness property of the real numbers.

(2) Give a precise definition of what it means for a sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ to converge to a real number L.

(3) If $a < x < b$ and $a < y < b$ show that $|x - y| < b - a$.

(4) Give an explicit example of a sequence of numbers which is bounded above, not bounded below, and which does not converge, and which does not diverge to $-\infty$.

(5) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{1}{n}$ converges.

(6) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{2n-1}{n+1}$ converges.

(7) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{n^2}{n+2}$ diverges to $+\infty$.

(8) Use the definition of convergence to show that the sequence given by $s_n = (-1)^n n$ does not converge.

(9) Suppose that $(s_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which converges. Show that the sequence $(s_n^2)_{n\in\mathbb{N}}$ also converges.

Midterm 1	
Overall	

Q1 Let A be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup(A)$ = least upper bound for A exists in $\mathbb{R}$.

Q2 For all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ for all $n \geq N$.

Q3
$$\left.\begin{array}{l} a < x < b \\ -b < -y < -a \end{array}\right\} \text{add}: \quad a-b < x-y < b-a \Rightarrow |x-y| < b-a$$

Q4 $\{0, -1, 0, -2, 0, -3, 0, -4, \dots\}$

Q5 show $s_n = \frac{1}{n} \to 0$ as $n \to \infty$.

Given $\epsilon > 0$, consider $\left|\frac{1}{n} - 0\right| < \epsilon \Leftrightarrow \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

so if we choose $N > \frac{1}{\epsilon}$, then $|s_n - 0| < \epsilon$ for all $n \geq N$, as required.

Q6 show $s_n = \frac{2n-1}{n+1} \to 2$ as $n \to \infty$.

Given $\epsilon > 0$, consider $\left|\frac{2n-1}{n+1} - 2\right| = \left|\frac{2n-1-2n-2}{n+1}\right| = \left|\frac{-3}{n+1}\right| = \frac{3}{n+1} < \frac{3}{n}$

so if we choose $N = \frac{3}{\epsilon}$ then $|s_n - L| < \epsilon$ for all $n \geq N$.

Q7 show $s_n = \frac{n^2}{n+2} \to +\infty$ as $n \to \infty$.

Given $M > 0$, want $\frac{n^2}{n+2} > M \Leftrightarrow n^2 > Mn + 2M \Leftrightarrow n^2 - Mn > 2M \Leftrightarrow \underbrace{n}_{\geq 2M}\underbrace{(n-M)}_{\geq 1} > 2M$

so if we choose $N = \max\{2M, M+1\} = 2M$, then $s_n \geq M$ for all $n \geq N$.

Q8 divergence: there is an $\epsilon > 0$ s.t. for all L for any $N \in \mathbb{N}$, there is an $n > N$ s.t. $|s_n - L| > \epsilon$.

Choose $\epsilon = \frac{1}{2}$, and $N = |L| + 1$, then $|s_n - L| \geq |s_n| - |L|$

as $s_n = (-1)^n n$, $|s_n| = n$, so for $n \geq N$, $|s_n - L| \geq |L| + 1 - |L| = 1 > \epsilon = \frac{1}{2}$, as required.

Q9 suppose $s_n \to L$, show $s_n^2 \to L^2$. $s_n \to L$ means for all $\epsilon > 0$, there is an N s.t. $|s_n - L| < \epsilon$ for all $n \geq N$.

Consider $|s_n^2 - L^2| = |(s_n - L)(s_n + L)|$ note: $|s_n - L| < \epsilon$, $|s_n + L| = |s_n - L + 2L| \leq |s_n - L| + 2|L| = \epsilon + 2|L|$, so $|(s_n - L)(s_n + L)| \leq \epsilon(\epsilon + 2|L|)$, so choose N such that $|s_n - L| < \epsilon/(2|L|+1) \cdot 2$

then $|s_n^2 - L^2| < \epsilon$ for all $n \geq N$.