could have constituted an advance.

These two 'class' definitions (ideal and real number) have no parallel since about 350 B.C. Eudoxus's (Fifth book of Euclid) definition of 'equal ratio' (of incommensurables) is in fact very near the Dedekind section (Eudoxus's equal $a : b$ and $c : d$ correspond each to the same 'class of rationals m/n; the two ratios are to be equal if the class of m/n's for which $ma < nb$ is identical with that for which $mc < nd$'.)

Turn now to another question: what is meant by a 'function'? I will digress (though with a purpose) to give some extracts from Forsyth's *Theory of Functions of a Complex Variable*; this is intended to make things easy for the beginner. (It was out of date when written (1893), but this is the sort of thing my generation had to go through. The fact that 'regularity' of a function of a complex variable is being explained at the same time adds unfairly to the general horror, but I should be sorry to deprive my readers of an intellectual treat.)

> All ordinary operations effected on a complex variable lead, as already remarked, to other complex variables; and any definite quantity, thus obtained by operations on z, is necessarily a function of z.
>
> But if a complex variable w is given as a complex function of x and y without any indication of its source, the question as to whether w is or is not a function of z requires a consideration of the general idea of functionality.
>
> It is convenient to postulate $u+iv$ as a form of the complex variable, where u and v are real. Since w is initially unrestricted in variation, we may so far regard the quantities u and v as independent and therefore as any functions of x and y, the elements involved in z. But more explicit expressions for these functions are neither assigned nor supposed.
>
> The earliest occurrence of the idea of functionality is in connection with functions of real variables; and then it is coextensive with the idea of dependence. Thus, if the value of X depends on that of x and on no other variable magnitude, it is customary to regard X as a function of x; and there is usually an implication that X is derived from x by some series of operations.
>
> A detailed knowledge of z determines x and y uniquely; hence the values of u and v may be considered as known and therefore also w. Thus the value of w is dependent on that of z, and is independent of the values of variables unconnected with z; therefore, with the foregoing view of functionality, w is a function of z.

It is, however, consistent with that view to regard as a complex function of the two independent elements from which z is constituted; and we are then led merely to the consideration of functions of two real independent variables with (possibly) imaginary coefficients.

Both of these aspects of the dependence of w on z require that z be regarded as a composite quantity involving two independent elements which can be considered separately. Our purpose, however, is to regard z as the most general form of algebraic variable and therefore as an irresoluble entity; so that, as this preliminary requirement in regard to z is unsatisfied, neither of these aspects can be adopted.

Suppose that w is regarded as a function of z in the sense that it can be constructed by definite operations on z regarded as an irresoluble magnitude, the quantities u and v arising subsequently to these operations by the separation of the real and imaginary parts when z is replaced by $x + iy$. It is thereby assumed that one series of operations is sufficient for the simultaneous construction of u and v, instead of one series for u and another series for v as in the general case of a complex function [above]. If this assumption be justified by the same forms resulting from the two different methods of construction, it follows that the two series of operations, which lead in the general case to u and to v, must be equivalent to the single series and must therefore be connected by conditions; that is, u and v as functions of x and y must have their functional forms related:

$$(1) \qquad \frac{\partial w}{\partial x} = \frac{1}{i}\frac{\partial w}{\partial y} = \frac{dw}{dy}$$

$$(2) \qquad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

These are necessary ... and sufficient ... relations between the functional forms of u and v.

The preceding determination of the necessary and sufficient conditions of functional dependence is based on the existence of a functional form; and yet that form is not essential, for, as already remarked, it disappears from the equations of condition. Now the postulation of such a form is equivalent to an assumption that the function can be numerically calculated for each particular value of the independent variable, though the immediate expression of the assumption has disappeared in the present case. Experience of functions of real variables shews that it is often more convenient to use their properties than to possess their numerical values. This experience is confirmed by what has preceded. The essential conditions of functional dependence are the equations (1) ...

Nowadays, of course, a function $y = y(x)$ means that there is a class of 'arguments' x, and to each x there is assigned 1 and only 1 'value' y. After some trivial explanations (or none?) we can be balder still, and say that a function is a class C of pairs (x, y) (order within the bracket counting), C being subject (only) to the condition that the x's of different pairs are different. (And a 'relation' R, 'x has the relation R to y', reduces *simply* to a class, which may be any class whatever, of ordered pairs.) Nowadays, again, the x's may be any sort of entities whatever, and so may the y's (e.g. classes, propositions). If we *want* to consider well-behaved functions, e.g. 'continuous' ones of a real variable, or Forsyth's $f(z)$, we *define* what being such a function means (2 lines for Forsyth's function), and 'consider' the class of functions so restricted. That is all. This clear daylight is now a matter of course, but it replaces an obscurity as of midnight.[1] The main step was taken by Dirichlet in 1837 (for functions of a real variable, the argument class consisting of some or all real numbers and the value class confined to real numbers). The complete emancipation of e.g. propositional functions belongs to the 1920's.

Suppose now, again to imagine a modified history, that the way out into daylight had been slightly delayed and pointed (as it so easily might have been) by the success of Dedekind's ideas. I will treat the idea of function, then, as derived from the Fermat theorem. (If this is rejected 'abolition' will be related instead to Fourier series or the differential equations of heat conduction.)

Consider now a function in which the argument class consists of the moments t of (historical) time and the value $f(t)$ for argument t is a state of the Universe (described in sufficient detail to record any happening of interest to anybody). If t_0 is the present date, $f(t)$, for $t < t_0$ is a description, or dictionary, of what *has* happened. Suppose now the dictionary transported back to an earlier time τ; then it contains a prediction of what is going to happen between times τ and t_0 This argument is clearly *relevant* to the issue of determination versus free-will and could reinforce any existing doubts. Doubts about free-will bear on the problem of moral responsibility and so (rightly or wrongly) on the problem of punishment. Wilder ideas have influenced reformers.

[1]The trouble was, of course, an obstinate feeling at the back of the mind that the value of a function 'ought' to be got from the argument by 'a series of operations'.