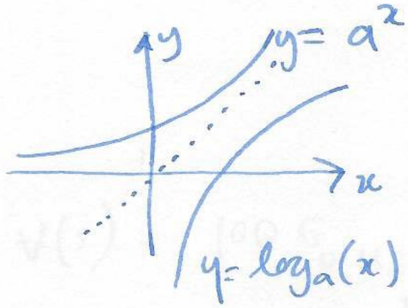
§4.3 Logarithms

recall: exponential function



one-to-one! (passes horizontal line test) so has an inverse we call $\log_a(x)$.

graph: reflect in $y=x$

$a^x : \mathbb{R} \to (0,\infty)$

$\log_a(x) : (0,\infty) \to \mathbb{R}$.

$$\log_a(x) = y \iff a^y = x.$$

inverse function property: $f(f^{-1}(x)) = x$ so $\log_a(a^x) = x$

$f^{-1}(f(x)) = x$ $\quad a^{\log_a(x)} = x$.

Examples $\log_a 1 = 0$ because $a^0 = 1$

$\log_a a = 1$ because $a^1 = a$

$\log_a a^2 = 2$ $\quad a^2 = a^2$.

$\log_a\left(\frac{1}{a}\right) = -1$ $\quad a^{-1} = \frac{1}{a}$

$\log_a(\sqrt{a}) = \frac{1}{2}$ $\quad a^{1/2} = \sqrt{a}$.

so $\log_2(4) = 2$ $\quad \log_3\left(\frac{1}{9}\right) = -2$ $\quad \log_4(2) = 1/2$ $\quad \log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$.

special logarithm $\log_e(x) = \ln(x)$ natural log inverse of e^x.

§4.4 Rules for Logarithms

reason $a^u = A$ $\quad a^v = B$.

$\log_a(AB) = \log_a(A) + \log_a(B)$ ← $AB = a^u a^v = a^{u+v}$

so $\log(AB) = \log(a^u a^v)$
$= \log(a^{u+v}) = u+v$
$= \log_a(A) + \log_a(B)$

$\log_a\left(\frac{A}{B}\right) = \log_a(A) - \log_b(B)$

$\log_a(A^B) = B\log_a(A)$. ← $\log_a(A^B) = \log_a((a^u)^B)$
$= \log_a(a^{uB}) = uB = B\log_a(A)$.

equations involving logs.

Examples:

$$\log_2(6x) = \log_2(6) + \log_2(x)$$

$$\log_5(x^2y^3) = \log_5(x^2) + \log_5(y^3) = 2\log_5(x) + 3\log_5(y).$$

$$\ln\left(\frac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln(\sqrt[3]{c}) = \ln(a) + \ln(b) - \tfrac{1}{3}\ln(c).$$

$$3\log x + \tfrac{1}{2}\log(x+1) = \log x^3 + \log\sqrt{x+1} = \log\left(x^3\sqrt{x+1}\right)$$

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change of base formula: $y = \log_b x$

take b^x of both side: $b^y = b^{\log_b(x)} = x$

take $\log_a$ of both sides: $\log_a(b^y) = \log_a(x)$

$$y\log_a(b) = \log_a(x)$$

$$\boxed{y = \frac{\log_a(x)}{\log_a(b)}}$$

in particular

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

Solving equations.

$2^x = 7$ $\quad \log_2(2^x) = \log_2(7)$ $\quad x = \log_2(7)$

$e^{3-2x} = 4$ $\quad \ln(e^{3-2x}) = \ln(4)$ $\quad 3-2x = \ln(4)$

$$2x = 3 - \ln(4)$$

$$x = \tfrac{3}{2} - \tfrac{1}{2}\ln(4)$$

$$= \tfrac{3}{2} - \ln(2).$$

$e^{2x} - e^x - 6 = 0$

quadratic in e^x!

$(e^x)^2 - e^x - 6 = (e^x - 3)(e^x + 2) = 0$

$e^x = 3$ $\quad x = \ln(3)$

$e^x = -2$ no solutions!