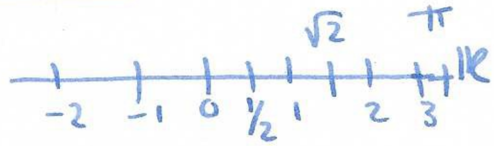
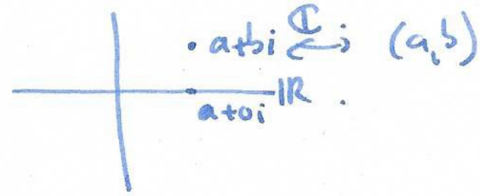
§3.5 Complex numbers

real numbers :



complex numbers: $i^2 = -1$

real part $\rightarrow a + bi \leftarrow$ imaginary part



Examples $\quad 3+4i \qquad \frac{1}{2} - \frac{1}{8}i$

$7i \qquad \pi i \qquad -4 = -4 + 0 \cdot i$

Operations : . addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$

. subtraction: $(a+bi) - (c+di) = (a-c) + (b-d)i$

Example $\quad 3+4i - (2-i) = 1+3i$

. multiplication: $(a+bi)(c+di) = ac + adi + bci + bd\underbrace{(i)^2}_{=-1}$

$= ac - bd + (ad+bc)i$

example :

$(3+4i)(2-i) = 3.2 - 3i + 8i - 4(i)^2$

$= 6 + 4 + 5i = 10 + 5i$

example : $\quad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \qquad i^{400} = (i^4)^{100} = (1)^{100} = 1$

$i^{401} = i^{400} \cdot i = i.$

. division $\quad$ the complex conjugate of $\underset{z}{a+bi}$ is $\underset{\bar{z}}{a-bi}$.

difference of two squares: $(a+bi)(a-bi) = a^2 + b^2$ (real!).

$$\frac{3+4i}{2-i}\,\frac{2+i}{2+i} = \frac{3.2 + 3i + 8i - 1}{4+1} = \frac{5+11i}{5}. = 1 + \frac{11}{5}i$$

Square roots of negative numbers: $\sqrt{-a} = \pm\sqrt{a}\, i \quad$ two roots!

check: $(\sqrt{a}\, i)^2 = a \cdot -1 = -a$

Solutions of quadratic equations

ax^2+bx+c has solutions $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Example ~~x^2+4x+5~~ x^2+9 $\quad x = \dfrac{0 \pm \sqrt{-4\times 9}}{2} = \sqrt{-9} = \pm 3i$

- $x^2+4x+5 \qquad x = \dfrac{-4 \pm \sqrt{4^2-4.5}}{2} = \dfrac{-4 \pm \sqrt{-4}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i.$
- $\underbrace{i}_{a}x^2+\underbrace{(2+i)}_{b}x+\underbrace{3-i}_{c}.$

Fact if x^2+4x+5 has complex solutions, then they are complex conjugates

§3.6 Fundamental theorem of algebra

Fundamental theorem of algebra: every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad a_n \in \mathbb{C}.$$

has at least one complex zero.

Complete factorization theorem

If $P(x)$ is a polynomial of degree ≥ 1, then there are complex numbers $a, c_1, \dots, c_n$ $(a \neq 0)$ s.t. $P(x) = a(x-c_1)(x-c_2)\cdots(x-c_n)$

Example $x^4+4x^2+4 = (x^2+2)^2 = (x+\sqrt{2}i)^2(x-\sqrt{2}i)^2.$

multiplicity of zeros if $P(x) = (x-1)^3(x+2)^2(x+3)^5.$

degree $= 3+2+5 = 10$ say $x=1$ occurs w/ multiplicity 3

$x=-2$ 2

$x=-3$ 5.

Zeros thm $P(x)$ degree n has n zeros, counted with multiplicity.

Fact if $P(x)$ has real coefficients, then if $P(z)=0$, then $P(\bar{z})=0$.

i.e. solutions come in complex conjugate pairs.