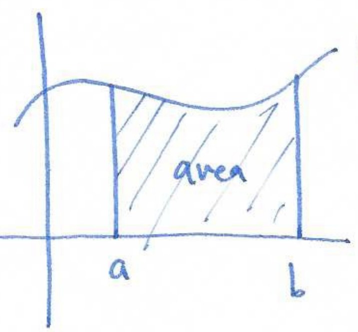
useful fact:

Thm If $f(x)$ is continuous on $[a,b]$ then all these approximations give the same limit as $N \to \infty$, which is the area under the curve.

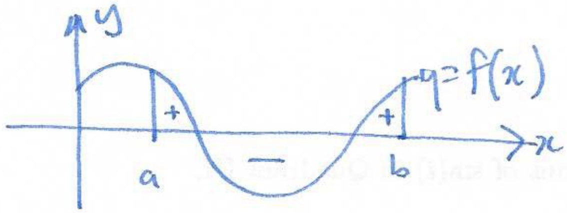
$$\text{area} = \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N$$

§5.2 Definite integral

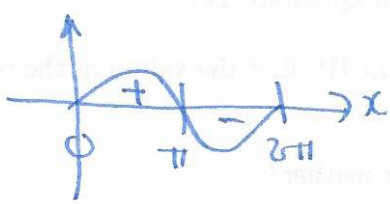


$$\int_a^b f(x)\,dx = \text{area under the curve } y=f(x) \text{ between } x=a \text{ and } x=b.$$

note: signed area:



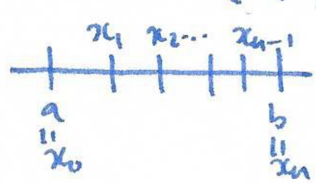
so $\displaystyle\int_0^{2\pi} \sin(x)\,dx =$



$= 0.$

Formal definition: Riemann sum $R(f,P,c)$

$P =$ partition of $[a,b]$



widths $\Delta x_i = x_i - x_{i-1}$

$C =$ choice of points $c_i \in [x_{i-1}, x_i]$

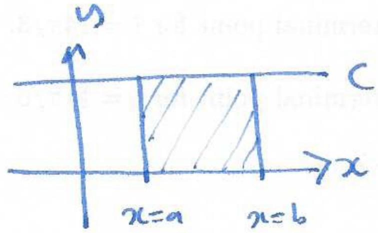
$$R(f,P,c) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i \qquad \|P\| = \max \Delta x_i$$

<u>Def^{n}</u> $$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f,P,c) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

when this limit exists, we say f is <u>integrable</u> over $[a,b]$.

<u>Useful properties</u>

$$\int_a^b c\,dx = c(b-a)$$



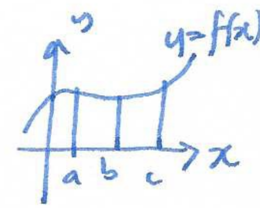
$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

reversing limits : $$\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx$$

0-length interval : $$\int_a^a f(x)\,dx = 0$$

adjacent intervals : $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

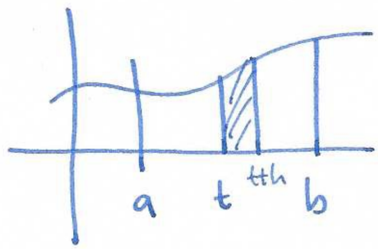


comparisons: if $f(x) \leq g(x)$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

§5.3 Fundamental theorem of calculus

<u>Th^{m}</u> (FTC①) suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an anti-derivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

intuition



consider $\int_a^t f(x)\,dx$

Q: what is rate of change wrt t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0}\frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0}\frac{\int_t^{t+h} f(x)\,dx}{h}$$

$$\underset{\text{approx}}{\approx} \frac{\text{area of rectangle } f(t)\times h}{h} = f(t)$$

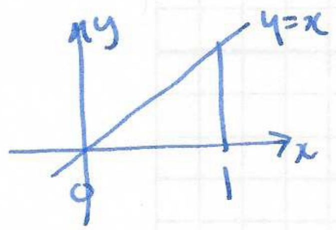
i.e. $\int_a^t f(x)\,dx$ is an anti-derivative for $f(x)$, so $\int_a^t f(x)\,dx = F(x) + c$

what is the constant? $t = a$: $\int_a^a f(x)\,dx = 0 = F(a) + c$

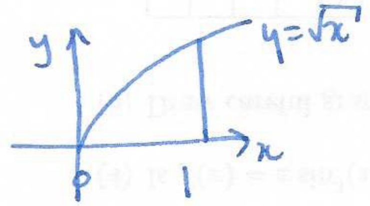
$$\Rightarrow c = -F(a)$$

so $\int_a^t f(x)\,dx = F(t) - F(a)$ $\square$

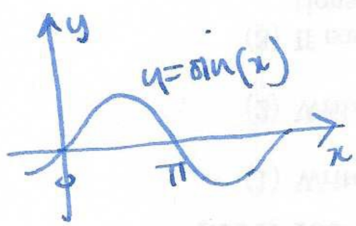
Examples



$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$



$$\int_0^1 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3} - 0 = \tfrac{2}{3}$$



$$\int_0^\pi \sin(x)\,dx = \left[-\cos(x)\right]_0^\pi = -\cos(\pi) + \cos(0)$$

$$= -(-1) + 1 = 2$$