Examples : find the general anti-derivative to $f(x) = \sin(4x)$.

guess : $\frac{d}{dx}\left(\cos(4x)\right) = -\sin(4x) \cdot 4$

so $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4} \cdot -\sin(4x) \cdot 4 = \sin(4x)$.

so $F(x) = -\frac{1}{4}\cos(4x) + c$

Notation : indefinite integral

$\int f(x)\,dx = F(x)+c$ means: $F(x)+c$ is the general anti-derivative for $f(x)$.

Thm $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c$ for $n \neq -1$.

Proof $\frac{d}{dx}\left(\frac{1}{n+1} x^{n+1}\right) = \frac{1}{n+1}(n+1)x^n = x^n$ $\square$.

Thm $\int \frac{1}{x}\,dx = \ln|x| + c$

Proof $(x>0)$ $\frac{d}{dx}\left(\ln(x)+c\right) = \frac{1}{x}$ $\square$.

Thm (sums and constant multiples)

$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int cf(x)dx = c\int f(x)\,dx$.

Warning : no product/ quotient / chain rule!

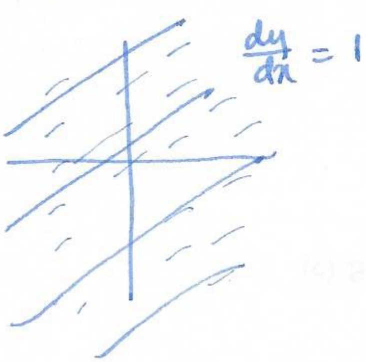
Useful integrals : $\int \sin(x)\,dx = -\cos(x)+c$ $\qquad \int e^x\,dx = e^x + c$

$\int \cos(x)\,dx = \sin(x)+c$

Example $\int x^2 + \frac{1}{x} + \sin(x)\,dx = \frac{1}{3}x^3 + \ln|x| - \cos(x) + c$

Alternate view

We can think of finding the indefinite integral as finding a function given its slope function, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (sometimes called an <u>initial condition</u>) then this gives a particular solution.



Example an object falls freely under gravity, so

acceleration: $a(t) = -g$ (constant)

velocity: $v(t)$ has $v'(t) = a(t)$

$$\frac{dv}{dt} = -g$$

has general solution $v(t) = -gt + c$

if the velocity at time $t=0$ v_0, then $v(0) = v_0 = c$

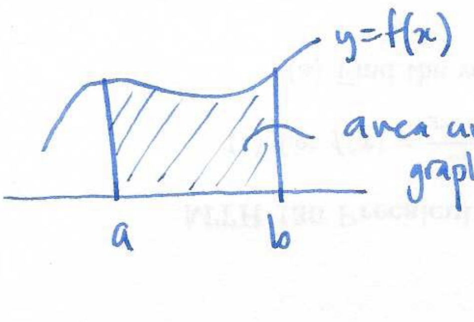
and so the particular solution is $v(t) = -gt + v_0$

position: $x(t)$, has $\frac{dx}{dt} = v(t) = -gt + v_0$

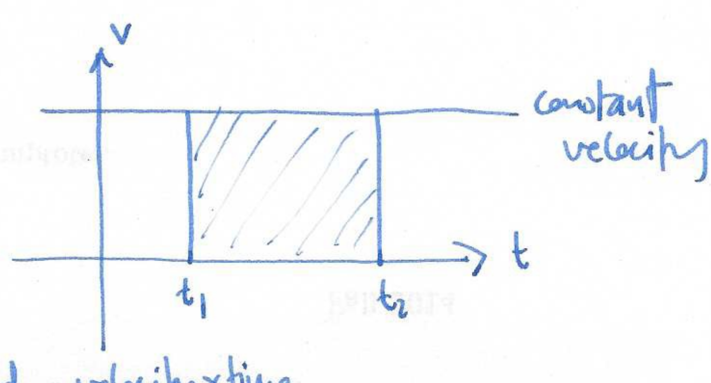
has general solution $x(t) = -\frac{1}{2}gt^2 + v_0 t + c$

if position at time $t=0$ is x_0, then $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$.

§5.1 Approximating areas

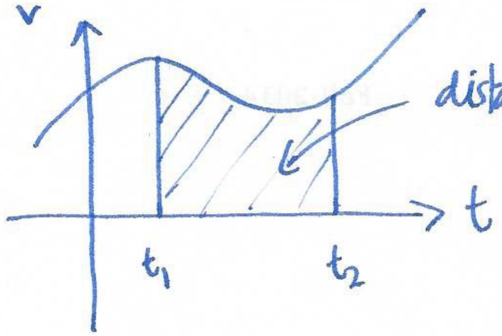


Example



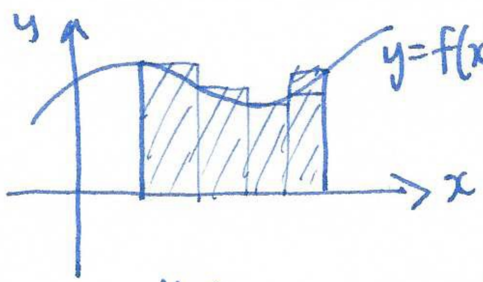
distance travelled = velocity × time
= area under the graph

non-constant velocity

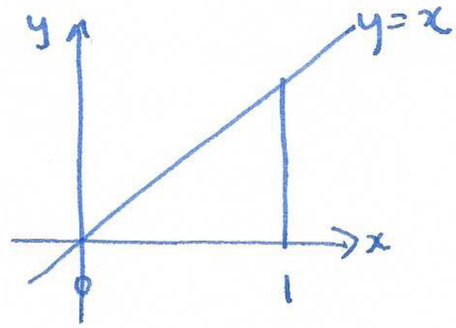


distance travelled = area under the graph

finding the area: approximate by rectangles



<u>Example</u>



find area under $y=x$ between 0 and 1 (answer $= \frac{1}{2}$)

approximate with 5 rectangles : area $\approx$ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$$= \frac{1}{5}f(0) + \frac{1}{5}f\left(\frac{1}{5}\right) + \frac{1}{5}f\left(\frac{2}{5}\right) + \frac{1}{5}f\left(\frac{3}{5}\right) + \frac{1}{5}f\left(\frac{4}{5}\right) = \sum_{i=0}^{4} \frac{1}{5} f\left(\frac{i}{5}\right)$$

$$= \frac{1}{5}\left(0 + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) = \frac{1}{25}(1+2+3+4) = \cancel{\sum_{i=0}^{4}} \frac{1}{25}\cdot 10 = \frac{10}{25} = 0.4 .$$

approximate with n rectangles:

$$\frac{1}{n}f(0) + \frac{1}{n}f\left(\frac{1}{n}\right) + \frac{1}{n}f\left(\frac{2}{n}\right) + \dots + \frac{1}{n}f\left(\frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right)$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n}\cdot\frac{i}{n}$$

$$= \frac{1}{n^2}(1+2+3+\dots+n-1) = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

<u>claim</u> : $1+2+3+\dots+n = \frac{1}{2}n(n+1)$

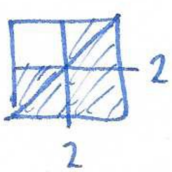
Proof ① induction: assume true for k:

$$s_k = 1+2+\cdots+k = \tfrac{1}{2}k(k+1)$$

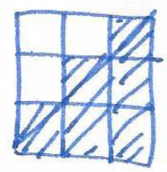
$$s_{k+1} = \underbrace{1+2+\cdots+k}_{\frac{1}{2}k(k+1)} + k+1 = \tfrac{1}{2}k(k+1) + k+1$$

$$= (k+1)\left(\tfrac{1}{2}k+1\right) = \tfrac{1}{2}(k+1)(k+2) \checkmark$$

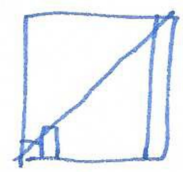
finally: check $k=1$: $1 = \frac{1}{2}1(2) = 1$ $\square$.

② 

$\frac{1}{2}(2)^2 + \frac{1}{2}\cdot 2$



$\frac{1}{2}(3)^2 + \frac{1}{2}(3)$



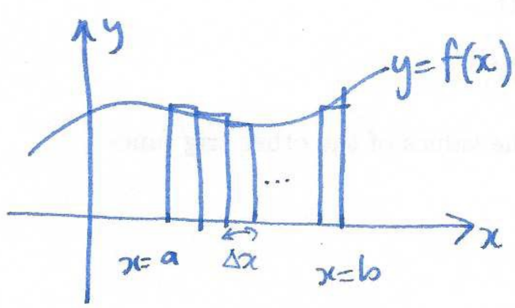
$\frac{1}{2}(n)^2 + \frac{1}{2}n$

$= \frac{1}{2}n(n+1)$ $\square$.

so approximate area is $\frac{1}{n^2}(1+2+\cdots+n-1) = \frac{1}{n^2}\frac{1}{2}(n-1)n$

$$= \frac{1}{2}\,\frac{n^2-n}{n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right) \to \frac{1}{2} \text{ as } n \to \infty.$$

Notation

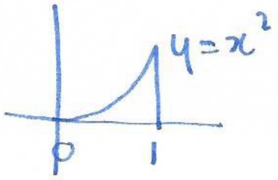


N rectangles of equal width then $\Delta x = \frac{b-a}{N}$

left endpoint rectangles $L_N = \sum_{j=0}^{N-1} f(a+j\Delta x)\Delta x$

right endpoint rectangles $R_N = \sum_{j=1}^{N} f(a+j\Delta x)\Delta x$

midpoint rectangles $M_N = \sum_{j=1}^{N} f\left(a+(j-\tfrac{1}{2})\Delta x\right)\Delta x$.



$1+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$