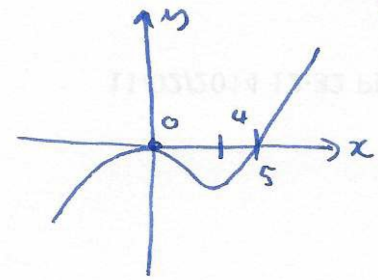
Example $f(x) = x^5 - 5x^4 = x^4(x-5)$



$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2$

(49)

critical points: $f'(x) = 0 \Rightarrow x = 0, 4$

2nd derivative test: $x=0 \quad f''(0) = 0$ no information

$x = 4 \quad f''(4) = 320 > 0 \Rightarrow$ local minimum.

at $x=0$ use first derivative test:

x		0	
$(x-4)$	$-$		$-$
x^3	$-$		$+$
$f'(x)$	$+$		$-$

$\Rightarrow$ local max.

§4.5 L'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable, and $f(a) = g(a) = 0$

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ provided this limit exists.

[Also works for $\lim\limits_{x\to a} f(x) = \pm\infty$, $\lim\limits_{x\to a} g(x) = \pm\infty$]

Warning ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be indeterminate form, can't use L'Hôpital on $\lim\limits_{x\to 0}\frac{1}{x}$.

Examples ① $\lim\limits_{x\to 1}\frac{x^3-1}{x-1} = \lim\limits_{x\to 1}\frac{3x^2}{1} = 3$

② $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{\sin x - 1} = \lim\limits_{x\to\frac{\pi}{2}}\frac{-2\cos x\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} -2\sin x = -2$

③ $\lim\limits_{x\to 0+} x\ln(x) = \lim\limits_{x\to 0+}\frac{\ln(x)}{1/x} = \lim\limits_{x\to 0+}\frac{1/x}{-x^{-2}} = \lim\limits_{x\to 0+} -x = 0.$

④ $\lim\limits_{x\to 0}\frac{e^x - x - 1}{\cos x - 1} = \lim\limits_{x\to 0}\frac{e^x-1}{-\sin x} = \lim\limits_{x\to 0}\frac{e^x}{-\cos x} = -e$

5 $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0$$

6 $\lim_{x\to 0} x^x = \lim_{x\to 0+} e^{x\ln(x)}$: note: e^x continuous so

$\Rightarrow e^{\lim_{x\to 0+} x\ln x} = \lim_{x\to 0+} e^{x\ln x}$

$\lim_{x\to 0} x\ln x = 0 \Rightarrow \lim_{x\to 0+} x^x = e^0 = 1$

Comparing growth rate of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x}$$

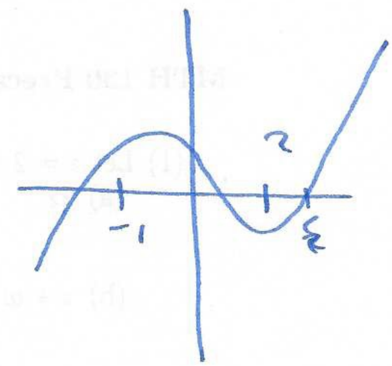
$= \lim_{x\to\infty} \frac{1}{8}x^{1/2} = \infty$, so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial.

Proof $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$.

§4.6 Graph sketching and asymptotes

<u>Example</u>: sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 = f(x)$



helpful info: · critical points
· sign of $f'(x)$
· sign of $f''(x)$

$f'(x) = x^2 - x - 2$
$f''(x) = 2x - 1$

critical points at $x = -1, +2$

$f''(-1) = -3 < 0$ max
$f''(2) = 3 > 0$ min

		-1		2	
$(x+1)$	$-$		$+$		$+$
$(x-2)$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

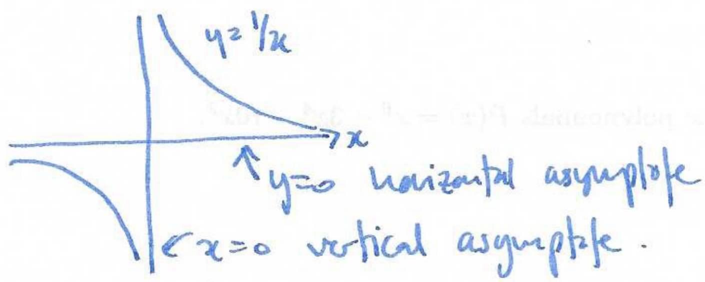
<u>Example</u>: $f(x) = (4x - x^2)e^x$
$f'(x) = (4 + 2x - x^2)e^x$
$f''(x) = (6 - x^2)e^x$

critical points $x = 1 \pm \sqrt{5}$

$1+\sqrt{5}$ local max $1-\sqrt{5}$ local min.

<u>Asymptotes</u>

Example $y = \frac{1}{x}$



<u>Def^{n}</u> $x = c$ is a <u>vertical asymptote</u> if $\lim_{x \to \pm c} f(x) = \pm\infty$

$y = c$ is a <u>horizontal asymptote</u> if $\lim_{x \to \pm\infty} f(x) = c$

<u>Observation</u> rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \le \deg q$.

<u>Example</u> $f(x) = \frac{x^2 + x + 1}{3x^2 + 2} \approx \frac{x^2}{3x^2} \Rightarrow \frac{1}{3}$ as $x \to \infty$.

<u>Example</u> sketch graph of $\frac{3x+2}{2x-4} = f(x)$

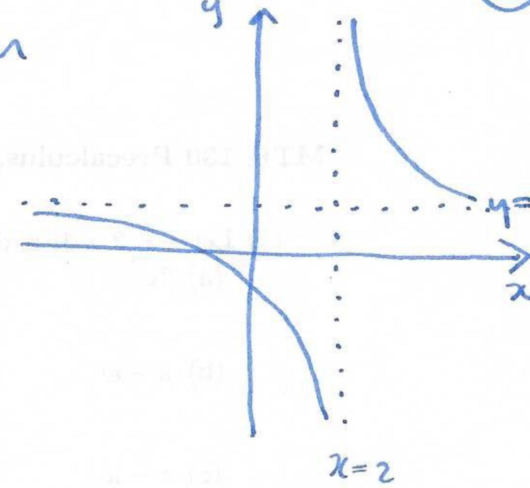
① find vertical asymptotes $\Leftrightarrow$ denominator zero $2x - 4 = 0 \Rightarrow x = 2$.

② find $f'(x) = \frac{(2x-4)\cdot 3 - (3x+2)\cdot 2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2} \Leftarrow$ always $-$ve $\Rightarrow$ (decreasing)

no critical points! (except $x = 2$).

③ $f''(x) = \dfrac{8}{(x-2)^3}$ +ve $x > 2$ concave up

−ve $x < 2$ concave down



④ horizontal asymptote $\dfrac{3x+2}{2x-4} \sim \dfrac{3x}{2x} = \dfrac{3}{2}$

behaviour near asymptote:

		$-2/3$		2	
$3x+2$	$-$		$+$		$+$
$2x-4$	$-$		$-$		$+$
$f(x)$	$+$		$-$		$+$

Example sketch graph of $\dfrac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

① vertical asymptotes: none

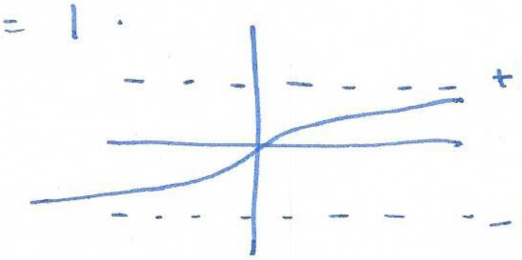
② find $f'(x) = \dfrac{\sqrt{x^2+1} - x\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{x^2+1} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2} > 0$ increasing, no critical points

③ $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\, 2x$ point of inflection at $x=0$

+ve for $x<0$

−ve for $x>0$

$\lim\limits_{x\to+\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x} = \lim\limits_{x\to\infty} \sqrt{1+1/x^2} = 1.$



$\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{-x}{\sqrt{x^2+1}} = -1.$

Example $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$

① vertical asymptotes at $x=-1,+1$

② $f'(x) = -(x^2-1)^{-2}\cdot 2x$ critical point at $x=0$

③ $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$ etc.