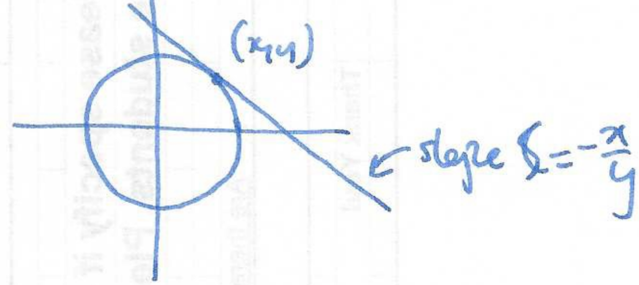
$$\frac{d}{dx}\left(x^2 + y(x)^2\right) = \frac{d}{dx}(1)$$

$$2x + 2y(x)\cdot y'(x) = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$




Example ① find tangent line to circle at $(3/5, 4/5)$

$$y' = -\frac{x}{y} \qquad \frac{-3/5}{4/5} = -\frac{3}{4} \qquad y - \frac{4}{5} = -\frac{3}{4}\left(x - \frac{3}{5}\right)$$

② find tangent line to $y^4 + xy = x^3 - x + 2$ at $(1,1)$

implicit differentiation: $4yy' + y + xy' = 3x^2 - 1$

at $x=1, y=1$ : $4y' + 1 + y' = 2 \quad 5y' = 1 \quad y' = \frac{1}{5} \quad y - 1 = \frac{1}{5}(x-1)$.

Inverse functions $y = f^{-1}(x) \Leftrightarrow f(y(x)) = x$

implicit differentiation : $f'(y(x))\cdot y'(x) = 1$

$$y'(x) = \frac{1}{f'(y(x))}$$

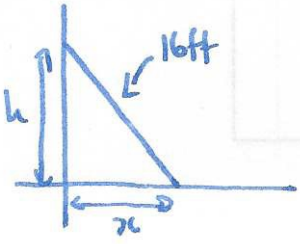
Example $y = \sinh^{-1}(x)$

$$\sinh(y) = x$$

$$\cosh(y)\, y' = 1 \qquad y' = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{1+x^2}}.$$

§ 3.11 Related rates

Example falling ladders




$x(t)$ distance of foot of ladder from wall
$h(t)$ height of top of ladder against wall.

Q : if $\frac{dx}{dt} = 3$ ft/s what is $\frac{dh}{dt}$ ?

note spoke at $t=0$, $x=5$.

true for all $t$ | true at some $t$

$x^2+h^2=16$ | $x(0)=5$

$\frac{dx}{dt}=3$

consider $x^2+h^2=16 \iff (x(t))^2+(h(t))^2=16$

differentiate wrt $t$ : $2x(t)\underbrace{\frac{dx}{dt}}_{3}+2h(t)\frac{dh}{dt}=0$

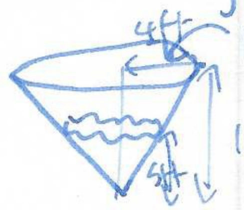
$\frac{dx}{dt}=3$ for all $t \Rightarrow$ $x(t)=3t+5$

$h(t)=\sqrt{16^2-(3t+5)^2}$

so $\frac{dh}{dt}=-\frac{x(t).3}{h(t)}=\frac{-3(5+3t)}{\sqrt{16^2-(3t+5)^2}}$

hit ground at $5+3t=16 \Rightarrow t=11/3$ $\frac{dh}{dt}(11/3)=\frac{-16}{\sqrt{16^2-16^2}}$ i.e. $\frac{dh}{dt}\to\infty$ as $t\to\frac{11}{3}$.

Example filling a conical tank




Q: how fast is the water rising when $h=5$ft?

water in : $\frac{dV}{dt}=10$ ft³/min

volume of cone : $V=\frac{1}{3}\pi h r^3$ need $r$ in terms of $h$ : $\frac{r}{h}=\frac{4}{10}$

i.e. $r=\frac{2}{5}h$

So $V=\frac{1}{3}\pi h\left(\frac{2}{5}h\right)^2=\frac{4}{75}\pi h^3$

$\frac{dV}{dt}=\frac{12}{75}\pi h^2\frac{dh}{dt}$ so when $h=5$ : $10=\frac{12}{75}\pi(5)^2\frac{dh}{dt}$

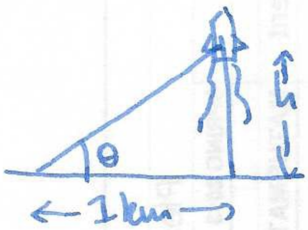
$\frac{dh}{dt}=\frac{10}{4\pi}$ ft/min.

<u>advice</u>: ① give things names

② write down relations between things and use implicit differentiation

③ plug in numbers if necessary.

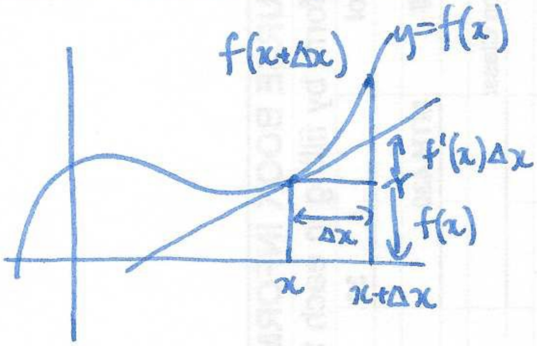
<u>Example</u>




if angle is $\theta = \frac{\pi}{3}$ and rate of change is $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min how fast is the rocket going?

$$\frac{h}{1} = \tan\theta \qquad \frac{dh}{dt} = \sec^2\theta \frac{dh}{dt} \qquad \frac{dh}{dt} = 10\sec^2\left(\frac{\pi}{3}\right)\frac{1}{2} \approx 1 \text{ km/min.}$$

## §4.1 Linear approximation




If $f(x)$ is differentiable at $x$ and $\Delta x$ is small, then

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

so change in $f$ is

$$\Delta f = f(x+\Delta x) - f(x)$$

$$\Delta f \approx f'(x)\Delta x$$

<u>Example</u> estimate $\sqrt{103}$

$$f(x) = \sqrt{x} = x^{1/2} \qquad f(100) = 10$$

$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(100) = \frac{1}{20}$$

so $\Delta f \approx f'(x)\Delta x = \frac{1}{20}\cdot 3$ so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$.

<u>Example</u> pizza size: you make an 18" pizza. If your diameter is accurate to $\pm 0.4$ in, how much pizza do you gain or lose?

$$A = \pi r^2 \quad 2r = D \qquad A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$$

$$A'(D) = \frac{2\pi D}{4} = \frac{\pi D}{2}$$

$$\Delta A \approx A'(18)\Delta D = \frac{1}{2}\pi 18 \times 0.4 \approx 11 \text{ in}^2.$$