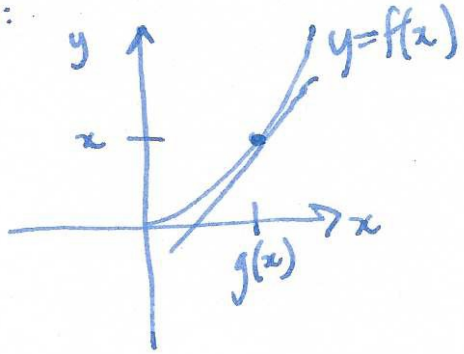
## §3.8 Derivatives of inverse functions

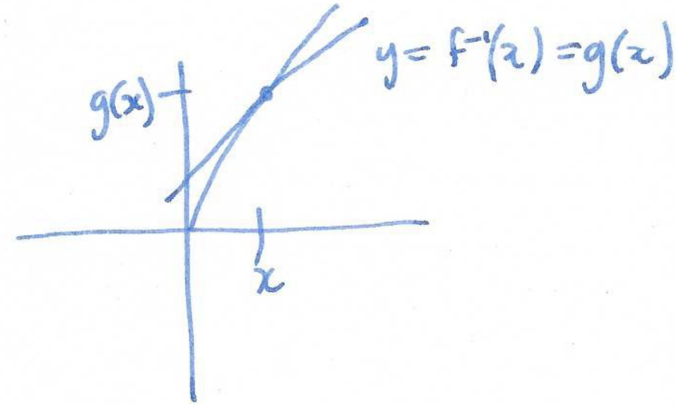
**Thm** suppose $f(x)$ differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

**recall:**




$\longrightarrow$ reflect in $y=x$




$\dfrac{1}{\text{slope}}$ at $g(x)$ $\longleftarrow$ slope at $x$: $g'(x)$

i.e. $\dfrac{1}{f'(g(x))}$

**Example** $f(x) = x^2$

$g(x) = f^{-1}(x) = \sqrt{x}$

$f'(x) = 2x \qquad g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}.$

**Thm** $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}} \qquad \dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$

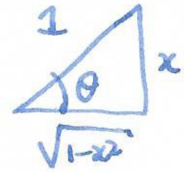
**Proof** (of $\sin^{-1}(x)$)

$f(x) = \sin(x) \qquad f'(x) = \cos(x)$

$g(x) = f^{-1}(x) = \sin^{-1}(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1}(x))}$$

recall: spose $\sin^{-1}(x) = \theta \iff \theta = \sin x$




want: $\cos\theta = \sqrt{1-x^2}$

so $$\frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}. \quad \square.$$

Thm: $$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1} \qquad \frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{x^2+1}$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}.$$

Example $$\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \frac{1}{1+(e^{2x})^2} \cdot 2e^{2x}$$

§3.9 Derivatives of exponentials and logs

recall: $f(x) = b^x$ then $f'(x) = m_b b^x$
$f(x) = e^x$ then $f'(x) = e^x$

Thm: $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{x\ln(b)}$ $\quad f'(x) = e^{x\ln(b)} \ln(b) = \ln(b)\, b^x$ $\square$

Example $f(x) = 3^{4x} = (3^4)^x \quad f'(x) = \ln(3^4)(3^4)^x = 4\ln(3)\, 3^{4x}.$

Thm: $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$

Proof $f(x) = e^x \qquad f^{-1}(x) = \ln(x) = g(x)$
$f'(x) = e^x$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} \quad \square.$$

Example ① $f(x) = \log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

$$f'(x) = \frac{1}{x\ln(b)}.$$

② $f(x) = x\ln(x)$

$$f'(x) = \ln(x) + x\cdot\frac{1}{x} = \ln(x) + 1$$

Hyperbolic trig functions $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\dfrac{d}{dx}(\sinh(x)) = \cosh(x)$ $\dfrac{d}{dx}(\cosh(x)) = \sinh(x)$. $\boxed{\cosh^2 x - \sinh^2 x = 1}$

check: $(\sinh(x))' = \left(\dfrac{e^x - e^{-x}}{2}\right)' = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$.

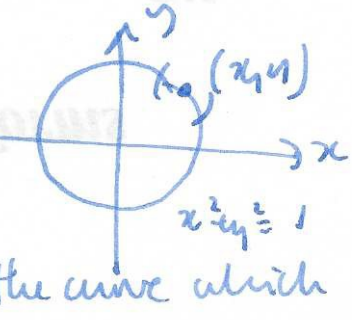
$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ so $\dfrac{d}{dx}(\tanh(x)) = \dfrac{1}{\cosh^2(x)}$ etc.....

inverse functions: $\dfrac{d}{dx}(\sinh^{-1}(x)) = \dfrac{1}{\sqrt{x^2+1}}$

$$\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{1-x^2}$$

§3.10 Implicit differentiation




explicit function: $f(x) = x^2 + x + 1$

implicit "function": $x^2 + y^2 = 1$

we get a local function near a point $(x,y)$ on the curve which we can think of as a function of $x$, i.e. $(x, f(x))$ or $(x, y(x))$.

differentiate $x^2 + y^2 = 1$ wrt $x$

$x^2 + (y(x))^2 = 1$ ← consider $y$ as $y(x)$ and use chain rule.