§3.6 Trigonometric functions

Thm $\frac{d}{dx}(\sin(x)) = \cos(x)$ $\quad \frac{d}{dx}(\cos(x)) = -\sin x$

Proof (for $\sin(x)$) $\qquad$ recall: $\sin(x+h) = \sin x \cos h + \cos x \sin h$

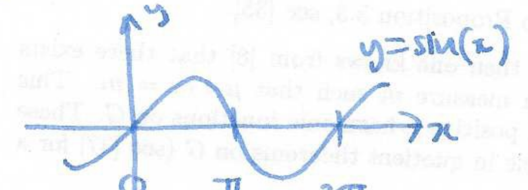
$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

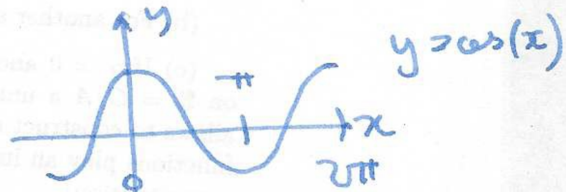
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} = \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \underbrace{\lim_{h\to x} \frac{\sin x}{x}}_{=1}$$

$$= \cos(x) \quad \square$$

Q: can this be right?





Example $f(x) = x\sin(x)$

$$f'(x) = x\cos(x) + \sin(x)$$

Thm $\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \tan x$

Proof (of $\tan x$)

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x . \cos x - \sin x . (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$\square$.

Example $\frac{d}{dx}(e^x \cos x) = e^x(-\sin x) + e^x \cos x$.

§3.7 The Chain rule

composition of functions: $f(g(x)) = (f \circ g)(x)$

Examples e^{4x}, $\sin^2(x)$, etc.

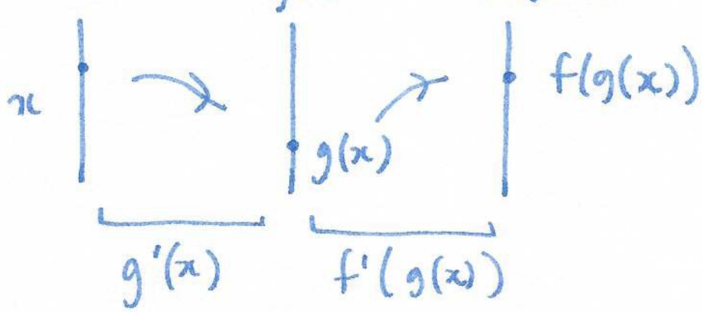
Thm Chain rule If f and g are differentiable functions, then $f \circ g$ is differentiable, and

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

mnemonic: $[f(g(x))]' = \text{outside}'(\text{inside}) \cdot (\text{inside})'$

note: $f(g(x))$

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$x \longmapsto g(x) \longmapsto f(g(x))$$



Examples

① $e^{4x} = f(g(x))$ where $f(x) = e^x$ $\quad f'(x) = e^x$
$\qquad g(x) = 4x \quad g'(x) = 4$

so $(e^{4x})' = f'(g(x)) \cdot g'(x) = e^{4x} \cdot 4$

② $\sin^2(x) = f(g(x))$ where $f(x) = x^2$ $\quad f'(x) = 2x$
$\qquad g(x) = \sin(x) \quad g'(x) = \cos(x)$

so $(\sin^2(x))' = f'(g(x))\, g'(x) = 2\sin(x)\cos(x)$.

③ $\sqrt{x^3+1}$, etc.

Alternative notation $f(g(x)) \leftrightarrow f(u) \quad u=g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$

mnemonic: "cancelling fractions"

Examples $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^3+1}}$.

Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{h}$$

[answer should be $f'(g(x))\,g'(x)$]

write this as: $\displaystyle\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \frac{g(x+h)-g(x)}{h}$ (*)

set $k = g(x+h)-g(x)$ as g is continuous $h\to 0 \Rightarrow k\to 0$

so $\displaystyle\lim_{k\to 0} \frac{f(g(x)+k)-f(g(x))}{k} = f'(g(x))$

so (*) $= f'(g(x))\,g'(x)$, as required. $\square$.

More examples $\dfrac{d}{dx}\left((g(x))^n\right) = n\,(g(x))^{n-1} g'(x)$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

$$\frac{d}{dx}\left(f(ax+b)\right) = af'(ax+b)$$