<u>Thm</u> (powers of x) if $f(x) = x^n$, then $f'(x) = nx^{n-1}$
(works for all $n \in \mathbb{R}$!).

<u>Examples</u> $\frac{d}{dx}(x^2) = 2x$ $\quad \frac{d}{dx}(x^3) = 3x^2$ $\quad \frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$ $\quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$\frac{d}{dx}(x^1) = x^0 = 1$ $\quad \frac{d}{dx}(x^0) = 0$

<u>Proof</u> Let $f(x) = x^n$, then $f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \cdots + h^n}$ all of these have a power of $h \geq 2$.

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $n! = 1 \cdot 2 \cdots n$

so $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots}{h} = \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots$

$= nx^{n-1}$ $\square$.

<u>Warning</u>: this rule works for polynomials only, not exponentials.

$f(x) = x^{100}$ polynomial! $\quad f(x) = 2^x$ <u>not</u> polynomial.

<u>Other useful rules</u>

<u>Thm</u> (linearity) If f and g are differentiable functions, then $f+g$ is differentiable, with $(f+g)' = f' + g'$

$$\Leftrightarrow \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

if k is a constant $\quad (kf)' = kf'$

$$\Leftrightarrow \frac{d}{dx}(kf) = k\frac{df}{dx}$$

Proof (follows from limit laws)

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$

$$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x)$$

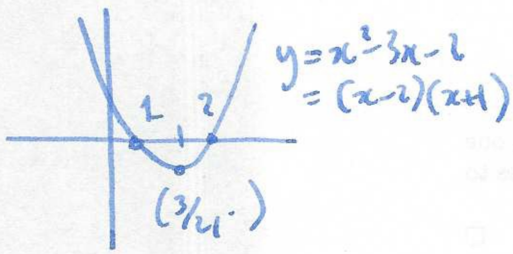
$\square$.

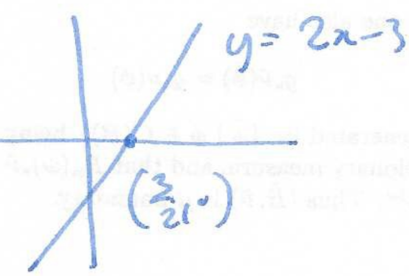
Example $f(x) = x^2 - 3x + 2$ find $f'(x)$

$$\frac{df}{dx} = \frac{d}{dx}\left(x^2 - 3x + 2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

$$= 2x - 3 + 0$$

graphs





Derivative of e^x

consider $f(x) = b^x$ $\quad b > 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} b^x\left(\frac{b^h - 1}{h}\right)$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}$$

→ doesn't depend on x!
assume this limit exists and call it m_b.

We have shown: for exponential functions the derivative is **proportional** to the value of the function, i.e. $f(x) = b^x$

$$f'(x) = m_b b^x$$

in particular, slope at $x=0$ is $f'(0) = m_b b^0 = m_b$

<u>recall</u> : e is defined to be the special number s.t. the slope of e^x at $x=0$ is equal to 1. Therefore if $\boxed{f(x)=e^x,\ f'(x)=e^x}$

$\leftrightarrow \boxed{\frac{d}{dx}(e^x) = e^x}$

<u>Example</u> $\frac{d}{dx}\left(7e^x+4x^2\right) = 7e^x + 8x$.

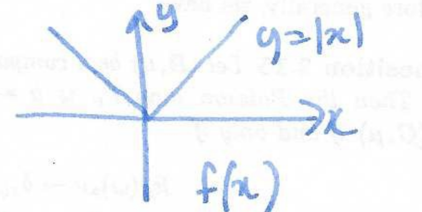
<u>Observation</u> : this shows that e^x is <u>not</u> a polynomial!

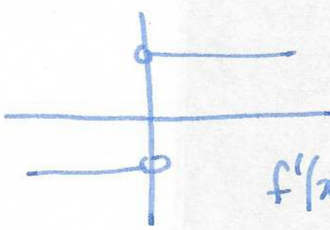
$\frac{d}{dx}(p(x)) \leftarrow$ degree goes down, eventually zero.

<u>Th^m</u> Differentiable $\Rightarrow$ continuous. <u>Warning</u> continuous $\not\Rightarrow$ differentiable

<u>Example</u> $f(x)=|x|$

continuous ✓





<u>claim</u> : not differentiable at $x=0$.

<u>check</u> : $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$ DNE at $x=0$.

when $x=0$: $f'(0) = \lim_{h\to 0} \frac{|h|}{h}$ DNE.

<u>local picture</u> if $f(x)$ differentiable at $x=c$, then if you look closely enough, the graph looks close to a straight line.

<u>Proof</u> (differentiable $\Rightarrow$ cts)

$f(x)$ differentiable at $x=c$ means $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists

(want to show $\lim_{h\to 0} f(c+h) = f(c)$)

consider $f(c+h)-f(c) = \frac{h\left(f(c+h)-f(c)\right)}{h}$

so $\lim_{h\to 0} f(c+h)-f(c) = \lim_{h\to 0} h \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ (products of limits)

$$= 0 \cdot f'(c) = 0 \quad \square.$$

§ 3.3 Product and quotient rules

new functions from old: $f(x)g(x)$ product, $\frac{f(x)}{g(x)}$ quotient.

Thm (Product rule) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

$$\frac{d}{dx}(fg) = \frac{df}{dx} g + f\frac{dg}{dx}$$

Warning $(fg)' \neq f'g'$!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x) = 1.x + x.1 = 2x$.

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = \frac{d}{dx}(3x^2).(x^2+1) + 3x^2.\frac{d}{dx}(x^2+1)$

$$= 6x.(x^2+1) + 3x^2.2x$$

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2).e^x + x^2.\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

Proof (of product rule) (assume f, g both differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x)g'(x) + f'(x)g(x) \quad \square.$$

Th^m <u>Quotient rule</u> (assume f, g differentiable at x, $g(x) \neq 0$)

Then $$\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - g'(x)\,f(x)}{(g(x))^2}$$

<u>Example</u> ① $$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x+1)'(x)}{(x+1)^2} = \frac{(x+1).1 - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

② $$\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t).(e^t)' - (e^t+t)'.e^t}{(e^t+t)^2}$$

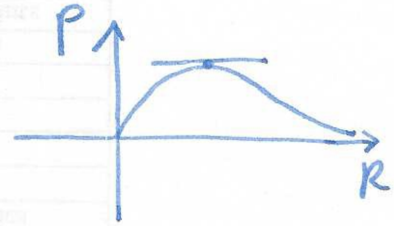
$$= \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2} = \frac{te^t - e^t}{(e^t+t)^2}$$

③ battery power

R resistance

internal resistance

power $P = \dfrac{V^2 R}{(r+R)^2}$



Q: when does the battery give maximal power?

<u>A</u> when $\dfrac{dP}{dr} = 0$

$P(R)$, V, r constant.

$$P = \frac{V^2 R}{(r+R)^2}$$

$$\frac{dP}{dR} = \frac{(r+R)^2.(V^2R)' - V^2R\left[(r+R)^2\right]'}{(r+R)^4} = \frac{(r+R)^2V^2 - V^2R(r+2rR+R^2)'}{(r+R)^4}$$

$$= \frac{V^2\left[(r+R)^2 - R(2r+2R)\right]}{(r+R)^4} = \frac{V^2\left[r+R\right]\left[r+R-2R\right]}{(r+R)^4} = \frac{V^2(r-R)}{(r+R^3)} = 0$$

when $r = R$.