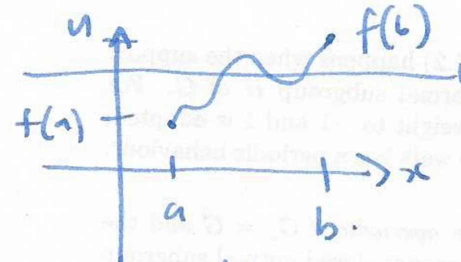
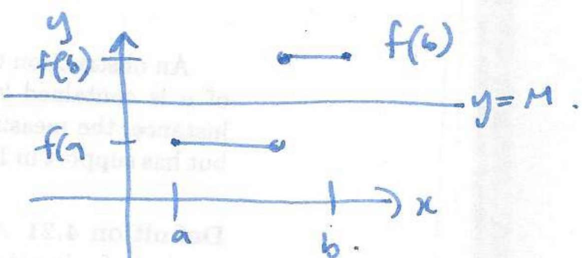
# § 2.8 Intermediate Value Theorem (IVT).



"continuous functions can't skip values"




continuous

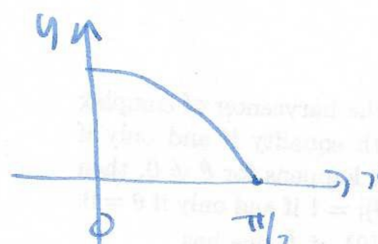



not continuous

Thm (Intermediate Value theorem IVT)

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$ then for any number $M$ between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = M$ □

Example show $\cos(x) = \frac{1}{4}$ has at least one sol$^{\underline{n}}$.




consider $[0, \frac{\pi}{2}]$ $\cos(0) = 1$

$\cos(\frac{\pi}{2}) = 0$

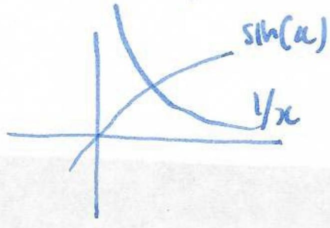
$0 \leq \frac{1}{4} \leq 1 \Rightarrow \exists\, c$ with $\cos(c) = \frac{1}{4}$.

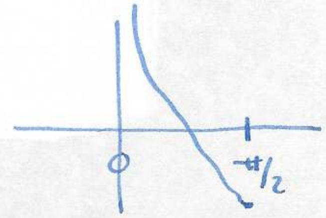
special case: finding zeros:

Corollary if $f(x)$ is continuous on $[a,b]$ and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ with $f(c) = 0$.

Bisection method: find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin(x)$



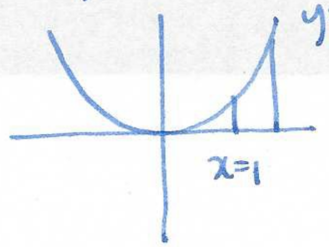

$f(0) = +\infty$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \approx -0.36$

midpoint: $\frac{\pi}{4}$ $f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566\ldots$ now continue with $[\frac{\pi}{4}, \frac{\pi}{2}]$. etc....

## §3.1 Definition of the derivative

<u>recall</u>: we can compute the average rate of change of a function over some interval $[x_0, x_1]$ : $\frac{f(x_1)-f(x_0)}{x_1 - x_0}$




<u>Q</u>: how do we compute the slope of the tangent line.

<u>idea</u>: look at average rate of change over small interval $[x, x+h]$ and take the limit as $h \to 0$

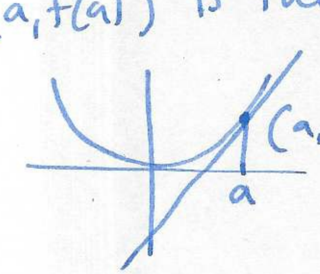
<u>Def<sup></sup></u> the slope of the tangent line at $x=a$ is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

<u>Notation</u>: also called the <u>derivative</u>, written $f'(a)$ or $\frac{df}{dx}(a)$
(Newton) (Liebnitz)

if this limit exists we say the function $f(x)$ is <u>differential</u> at $x=a$

<u>note</u>: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ same as $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

<u>Def<sup></sup></u> The tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$




the equation for this line is $y - y_0 = m(x - x_0)$

i.e. $y - f(a) = f'(a)(x-a)$

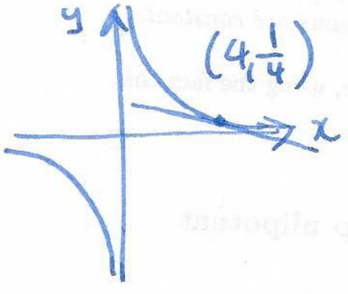
<u>Example</u> find tangent line to $y = x^2$ at $x=1$

$(x, f(x))$ is $(1,1)$ slope $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{1+2h+h^2-1}{h} = \lim_{h \to 0} 2+h = 2.$

so equation of tangent line is $y-1 = 2(x-1)$

Example find slope of tangent line to $f(x)=\frac{1}{x}$ at $x=4$.




slope $f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h}-\frac{1}{4}}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left(\frac{4-(4+h)}{(4+h)4}\right) = \lim_{h\to 0} \frac{4-4-h}{4h(4+h)} = \lim_{h\to 0} \frac{-h}{4h(4+h)}$

$= \lim_{h\to 0} \frac{-1}{4(4+h)} = -\frac{1}{16}$ $\quad y-\frac{1}{4} = -\frac{1}{16}(x-4)$.

Example straight line $y=mx+b$

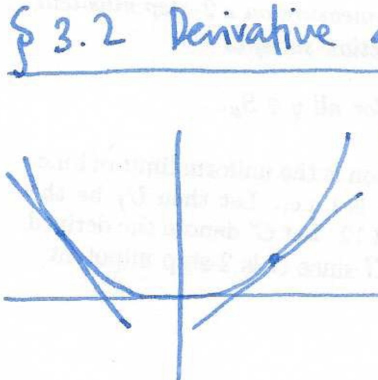
find slope at $x=a$: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$= \lim_{h\to 0} \frac{m(a+h)+b-(ma+b)}{h} = \lim_{h\to 0} \frac{\cancel{ma}+mh+\cancel{b}-\cancel{ma}-\cancel{b}}{h}$

$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$.

Observation if $f(x)=b$ (constant), then $f'(x)=0$ for all $x$.

§ 3.2 Derivative as a function



$y=x^2 \quad f:\mathbb{R}\to\mathbb{R}$

$x \mapsto x^2$

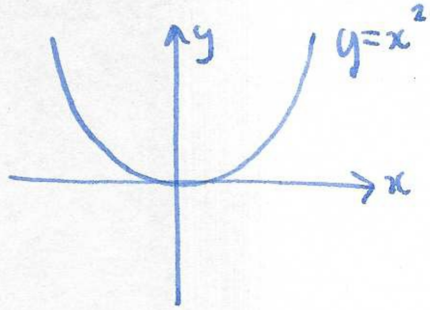
at each input point $x$, there is a tangent line with a slope, which is a number.

therefore we can define a function

$x \longmapsto$ slope of tangent line at $x$

notation: we call this function $f'(x)$ or "the derivative of $f$".

<u>Example</u>



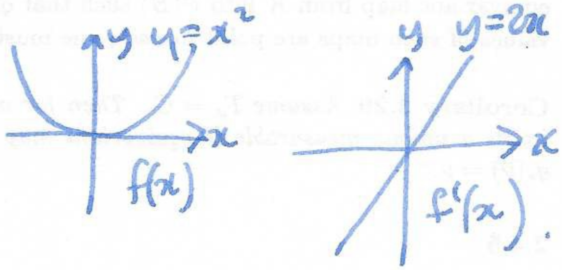

slope at $x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if $f(x) = x^2$ then $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2}{h} = \lim\limits_{h\to 0} 2x+h = 2x$

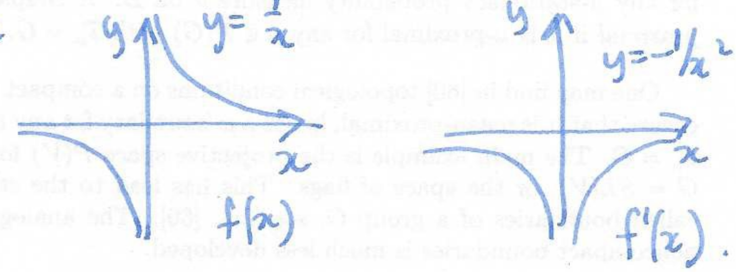
<u>summary</u>: if $f(x) = x^2$, then $f'(x) = 2x$




<u>Example</u> $f(x) = \dfrac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\,\frac{x-(x+h)}{(x+h)x}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$




<u>Remarks</u>

① functions: $f$: domain $\to$ range

e.g. $f: \mathbb{R} \to \mathbb{R}$

derivative : (differentiable) functions $\to$ functions

$f(x) \longmapsto f'(x)$ or $\dfrac{df}{dx}$

<u>Warning</u>: not all functions differentiable!

② "calculus" means rules for doing calculations — we don't have to explicitly compute limits all the time.

<u>Example</u> $f(x) = x^3$ $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim\limits_{h\to 0} 3x^2+3xh+h^2 = 3x^2$