Examples

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} \qquad \left.\begin{array}{l} \tan(\pi/2) = \dfrac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0 \\ \sec(\pi/2) = \dfrac{1}{\cos(\pi/2)} = 1/0 \end{array}\right\} \text{undefined.}$$

but
$$\frac{\tan(x)}{\sec x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x \quad (\text{if } \cos x \neq 0)$$

so
$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$$

Example $\lim_{x\to 0} \frac{1}{x-1} - \frac{2}{x^2-1}$ $\quad \frac{1}{0} - \frac{1}{0}$ undefined.

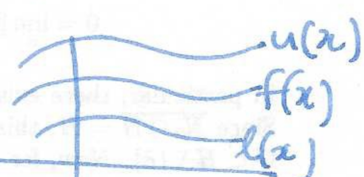
$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{1}{x+1} \quad (x\neq 1) \quad \lim_{x\to 0} \frac{1}{x+1} = \frac{1}{2}.$$

§2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin(x)}{x}$? substitute $x=0$: get $\frac{0}{0}$, undefined.

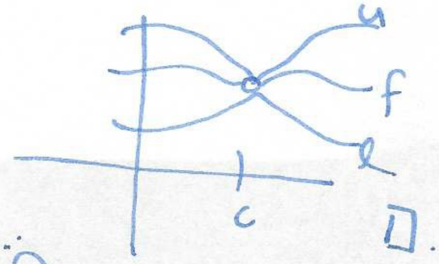
A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try $x = 0.1, 0.01, 0.001, \ldots$)

squeeze theorem : suppose $l(x) \leq f(x) \leq u(x)$



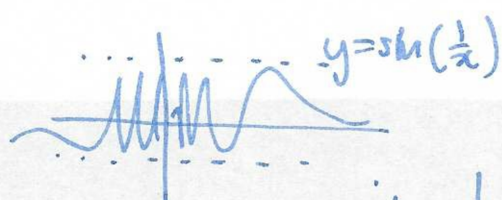
Thm suppose $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$

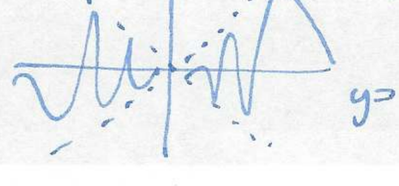
then $\lim_{x\to c} f(x) = L$.



□.

Example $\lim_{x\to 0} x\sin(\frac{1}{x})$





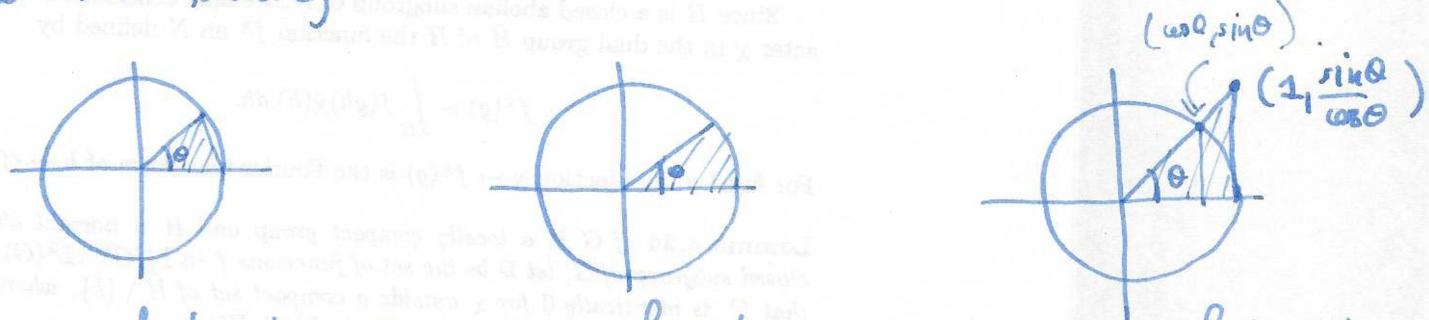
$$-|x| \leq x\sin\left(\tfrac{1}{x}\right) \leq |x| \qquad \lim_{x\to 0} |x| = 0 \qquad \lim_{x\to 0} -|x| = 0$$

so $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

<u>Thm</u> $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

<u>Proof</u> (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$,

consider the following 3 areas:



area of triangle $\leq$ area of sector $\leq$ area of triangle

$\frac{1}{2}bh \qquad \pi r^2 \cdot \frac{\theta}{2\pi} \qquad \frac{1}{2}bh$

$$\underbrace{\frac{1}{2}1.\sin\theta \leq \frac{\theta}{2}}_{\frac{\sin\theta}{\theta} \leq 1} \underbrace{\leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}}_{\cos\theta \leq \frac{\sin\theta}{\theta}}$$

$\Rightarrow \cos\theta \leq \frac{\sin\theta}{\theta} \leq 1 \qquad \lim_{\theta\to 0} 1 = 1 \qquad \lim_{\theta\to 0} \cos\theta = 1$

so squeeze theorem $\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ $\square$.

Example

① $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$ know $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

can write this as $3x = \theta$: $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{2\theta/3} = \lim\limits_{\theta\to 0} \dfrac{3}{2}\dfrac{\sin\theta}{\theta}$

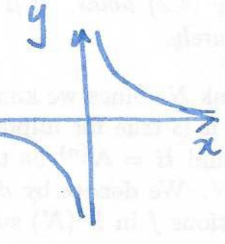
$= \dfrac{3}{2} \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = \dfrac{3}{2}.$

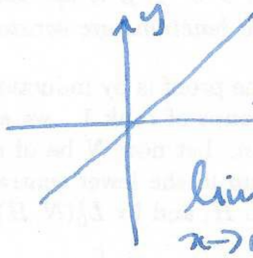
② $\lim\limits_{t\to 0} \dfrac{1-\cos t}{\sin t} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{t}\,\dfrac{t}{\sin t}$ (*)

$\lim\limits_{t\to 0} \dfrac{1-\cos t}{t} = 0$ $\lim\limits_{t\to 0} \dfrac{t}{\sin t} = \dfrac{1}{\lim\limits_{t\to 0} \dfrac{\sin t}{t}} = \dfrac{1}{1} = 1.$ (*) : $0 \cdot 1 = 1.$

§2.7 Limits at infinity

key observation: $\lim\limits_{x\to 0} \dfrac{1}{x} = 0$





$\lim\limits_{x\to\infty} x = \infty.$

Examples

① $\lim\limits_{x\to\infty} \dfrac{3x}{2x-1} = \lim\limits_{x\to\infty} \dfrac{3}{2-1/x} = \dfrac{3}{2}$

② $\lim\limits_{x\to\infty} \dfrac{x^2+x}{x-3} = \lim\limits_{x\to\infty} \dfrac{x+1}{1-3/x} = \infty.$

③ $\lim\limits_{x\to\infty} \dfrac{1}{x} - \dfrac{2}{3x+1} = \lim\limits_{x\to\infty} \dfrac{1}{x} - \lim\limits_{x\to\infty} \dfrac{2}{3x+1} = 0 - 0 = 0$

④ $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{4x+1} = \dfrac{\sqrt{2+1/x^2}}{4+1} = \dfrac{\sqrt{2}}{4}.$