§2.3 Basic limit laws

Example $\lim_{x\to 0} 2x+2 = \lim_{x\to 0} 2x + \lim_{x\to 0} 2 = 0+2 = 2.$

Thm assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist and are finite.

Then

1) sums: $\lim_{x\to c} \left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} k f(x) = k \lim_{x\to c} f(x)$ k constant (does not depend on x)

3) products: $\lim_{x\to c} \left(f(x)g(x)\right) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

4) quotients: $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ as long as $\lim_{x\to c} g(x) \neq 0$.

Warning: these rules won't work if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ DNE.

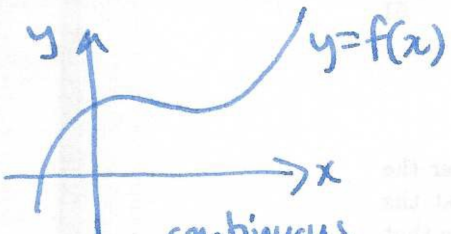
Examples:

$$\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 3.3 = 9$$

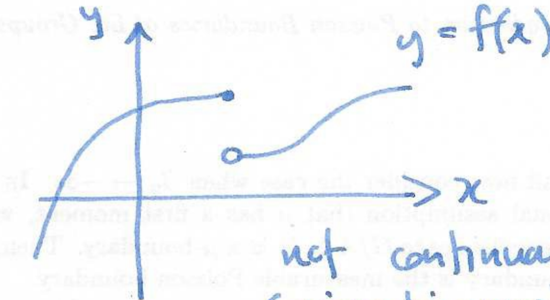
$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\left(\lim_{t\to 2} t+5\right)}{\left(\lim_{t\to 2} 3t\right)} = \frac{7}{6}$$

§2.4 Limits and continuity

Example



continuous

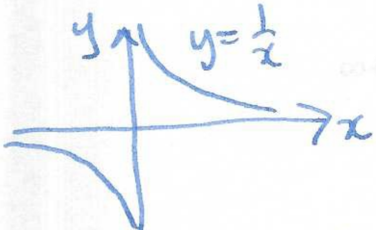


not continuous (discontinuous)

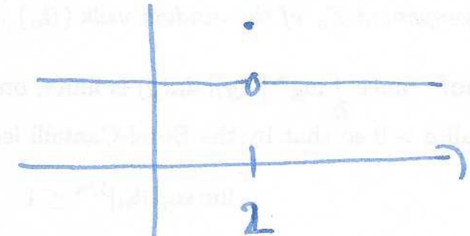
Def^{n} We say $f(x)$ is continuous at $x=c$ if $\lim\limits_{x\to c} f(x) = f(c)$

if the limit does not exist, or is not equal to $f(c)$, or is infinite, then $f(x)$ is not continuous at $x=c$.

Example



$f(x)=\frac{1}{x}$ not continuous at $x=0$

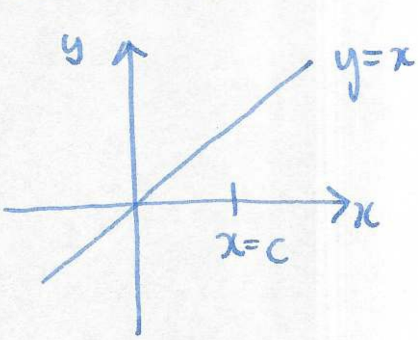


$$f(x)=\begin{cases} 1 & x\neq 2 \\ 2 & x=2 \end{cases}$$

not continuous at $x=2$

Example

show $f(x)=x$ is continuous



$f(c)=c$

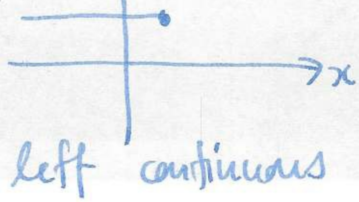
want to show $\lim\limits_{x\to c} f(x) = f(c)$

follows from limit law: $\lim\limits_{x\to c} x = c$

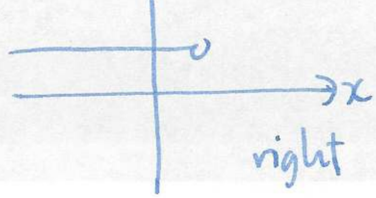
Def^{n} $f(x)$ is left continuous at $x=c$ if $\lim\limits_{x\to c-} f(x) = f(c)$

right continuous at $x=c$ if $\lim\limits_{x\to c+} f(x) = f(c)$

Examples



left continuous



right continuous

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$.

Building continuous functions

Thm 0 $\quad f(x)=k$, $f(x)=x$ are continuous.

Thm 1 Suppose that $f(x)$ and $g(x)$ are both continuous at $x=c$. Then the following functions are continuous at $x=c$

1) $f(x)+g(x)$

2) $kf(x)$ for any constant k

3) $f(x)g(x)$

4) $\dfrac{f(x)}{g(x)}$ if $g(c)\neq 0$

Proof: these follow directly from the limit laws.

check 1) $f(x)$ cts means $\lim\limits_{x\to c} f(x) = f(c)$

$g(x)$ cts mean $\lim\limits_{x\to c} g(x) = g(c)$

so $\lim\limits_{x\to c}(f(x)+g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = f(c)+g(c)$

as required $\square$

Thm 2 Polynomials are continuous $\quad P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$

Rational functions $\dfrac{p(x)}{q(x)}$ are continuous, except where $q(x)=0$

Proof $f(x)=x$ is continuous

so $f(x)f(x) = x.x = x^2$ is continuous (products)

similarly x^n is continuous

so $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is cts (multiply by constant and addition)

so $\dfrac{p(x)}{q(x)}$ is continuous (quotients) when $q(x) \neq 0$. $\square$.

<u>Useful facts</u>

<u>Thm 3</u>. • $\sin(x)$, $\cos(x)$ are continuous

• b^x is continuous $(b > 0)$

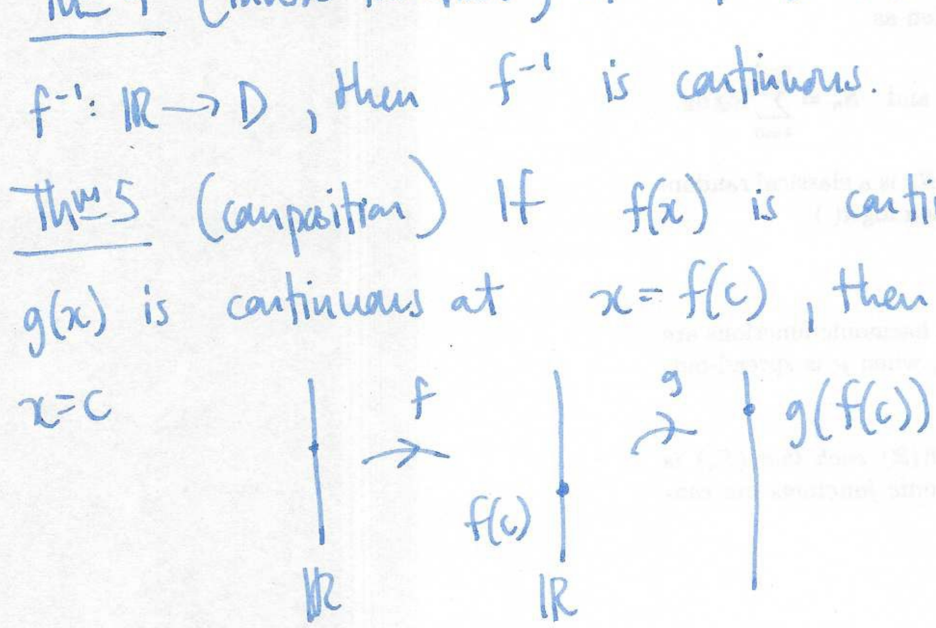
• $\log_b(x)$ is continuous

• $x^{1/n}$ is continuous

~~these~~ (combinations of these are sometimes called elementary functions)

<u>Thm 4</u> (inverse functions) If $f: D \to \mathbb{R}$ is continuous, with inverse $f^{-1}: \mathbb{R} \to D$, then f^{-1} is continuous.

<u>Thm 5</u> (composition) If $f(x)$ is continuous at $x = c$, and $g(x)$ is continuous at $x = f(c)$, then $g(f(x))$ is continuous at



<u>Examples</u> $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2 + x + 1}}$ continuous at $x = 1$

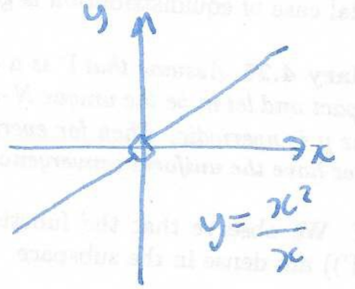
Q: where is $f(x) = \dfrac{x^2}{\sin(x)}$ cts?

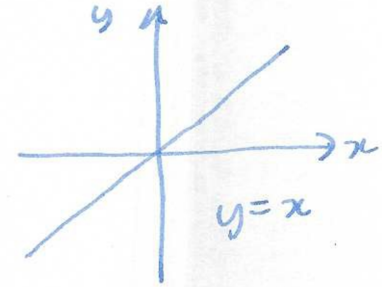
§2.5 Evaluating limits algebraically

Example $\dfrac{x^2}{x}$ undefined at $x=0$: $\dfrac{0}{0}$ indeterminate form

but $\displaystyle\lim_{x\to 0}\frac{x^2}{x}$ does not depend on value at $x=0$.

$\dfrac{x^2}{x} = x$ for $x \neq 0$





so $\displaystyle\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0$

Indeterminate forms : $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, 0^0

note : $\dfrac{1}{0}$ not indeterminate, gives limit $\pm\infty$ or DNE.

Example $\displaystyle\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}$ $\quad x=3$: $\dfrac{9-12+3}{9+3-12} = \dfrac{0}{0}$

factor : $\dfrac{(x-3)(x+1)}{(x-3)(x+4)} = \dfrac{x-1}{x+4}$ $(x\neq 3)$

so $\displaystyle\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12} = \lim_{x\to 3}\frac{x-1}{x+4} = \frac{2}{7}$

Example $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ $\qquad \dfrac{2-2}{4-4} = \dfrac{0}{0}$

$\dfrac{\sqrt{x}-2}{x-4} = \dfrac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \dfrac{1}{\sqrt{x}+2}$ so $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{4}$

Example $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\tan x}{\sec x}$ $\qquad$ **Example** $\displaystyle\lim_{x\to 1}\frac{1}{x-1}-\frac{2}{x^2-1}$