## §1.5 Inverse functions

recall: $f: X \to Y$
domain range
$x \mapsto f(x)$

want: the inverse function should be the reverse of this

$X \leftarrow Y : f^{-1}$

$x \leftarrow\!\shortmid f(x)$

problem: the inverse is often not a function

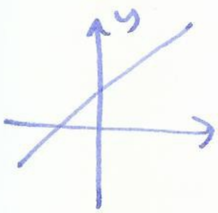
$a, b \to f(a) = f(b)$ : suppose there is $a \neq b$ s.t. $f(a) = f(b)$.
what is $f^{-1}(f(a))$?

Q: when does a function have an inverse?

A: when it passes the horizontal line test (one-to-one / injective)

$\Leftrightarrow$ for each value $c \in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$.

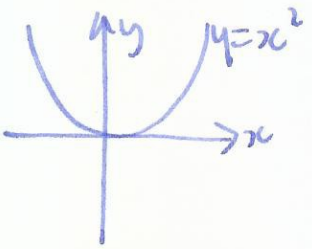
**Example** $y = x+1$




Q: how do we find a formula for the inverse?

A: ① write down $y = f(x)$
② solve for $x$ in term of $y$, i.e. $x = g(y)$
③ $f^{-1}(x) = g(x)$.
④ check!

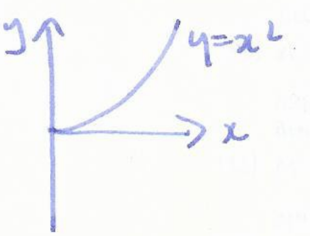
**bad example** $f(x) = x^2$




problem: doesn't pass horizontal line test

fix: restrict domain, consider $f: [0,\infty) \to [0,\infty)$

$x \mapsto x^2$.

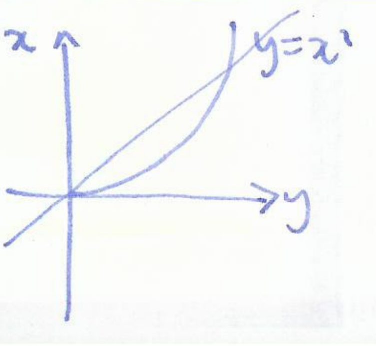



this does pass the horizontal line test
so there is an inverse we call $\sqrt{x}$

$f^{-1}: [0,\infty) \to [0,\infty)$.

$x \mapsto \sqrt{x}$.

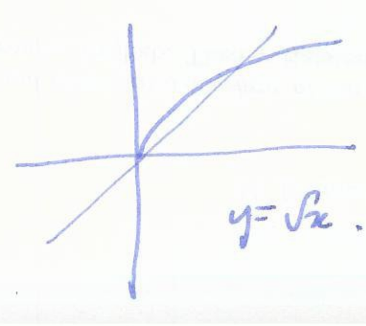
How to draw the graph of the inverse:




A $\boxed{\text{reflect in } y = x}$

reason: graph of $f$: $(x, f(x))$

$f^{-1}$: $(x, f^{-1}(x))$



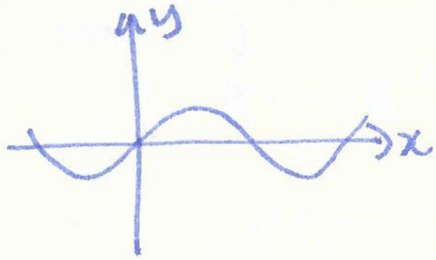
$y = f(x) \Leftrightarrow f^{-1}(y) = x$ $\qquad$ $(x, f(x)) \Leftrightarrow (f^{-1}(y), y)$

swap

$(y, f^{-1}(y))$ can relabel dummy variable.

## Inverse trig functions

$y = \sin(x)$



problem: not one-to-one

fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

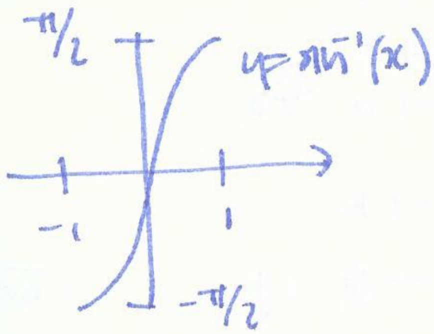
$\sin^{-1}(x) : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\shortparallel$

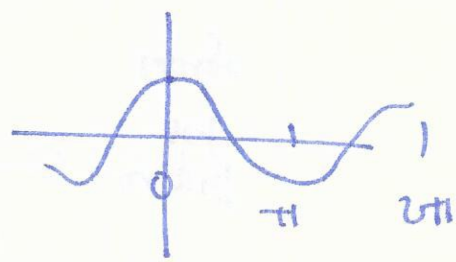
$\text{asin}(x)$

$\shortparallel$

$\text{arcsin}(x)$




similarly $y = \cos(x)$




restrict domain to $[0, \pi]$

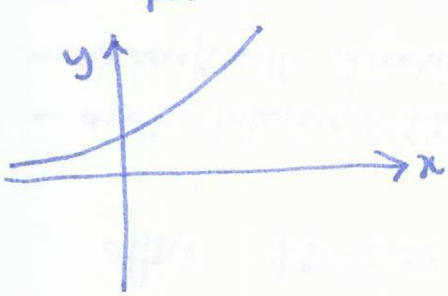
$\cos(x) : [0, \pi] \to [-1, 1]$

$\cos^{-1}(x) : [-1, 1] \to [0, \pi]$

## §1.6 Exponential and logarithm functions

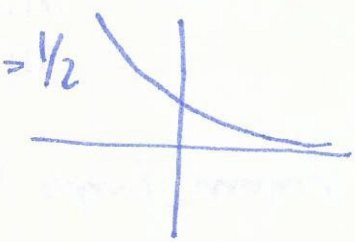
Example $x \mapsto 2^x$

| $x$ | $-2$ | $1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $1/4$ | $1/2$ | $1$ | $2$ | $4$ | $8$ |



can use any positive number instead of 2

$f(x) = b^x \quad (b > 0)$ $\qquad$ $b = 1/2$



useful properties

- positive $b^x > 0$ for all $x$.
- $b^x$ increasing if $b > 1$
- decreasing if $0 < b < 1$

$b^x$ grows faster than any polynomial $x^n$.

<u>exponent rules</u>:

$$b^0 = 1, \quad b^x b^y = b^{x+y}, \quad b^{-x} = \frac{1}{b^x}, \quad \frac{b^x}{b^y} = b^{x-y}$$

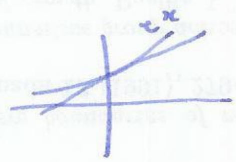
$$(b^x)^y = b^{xy}, \quad b^{1/n} = \sqrt[n]{b}.$$

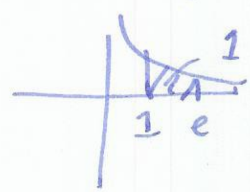
- there is a special exponential function $e^x$ $\quad e = 2.71828\ldots$

<u>key properties</u>

① $e$ is the unique number such that $e^x$ has slope 1 at $x=0$

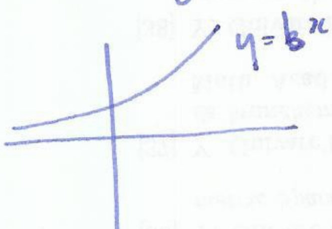
② $e$ is the unique number s.t. the area under $y = 1/x$ between 1 and $e$ has area 1

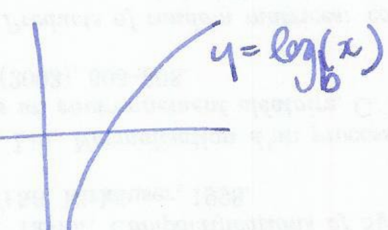



<u>Logarithms</u>

The logarithm is the <u>inverse function</u> for the exponential function




the special logarithm with base $b = e$ is called the <u>natural logarithm</u> $\ln(x)$.

- inverse function properties: $f^{-1}(f(x)) = x = f(f^{-1}(x))$

$$b^{\log_b(x)} = x = \log_b(b^x)$$

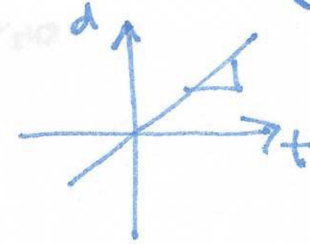
- logarithm rules: $\log_b(1) = 0 \quad \log_b(b) = 1$

$$\log_b(st) = \log_b(s) + \log_b(t) \qquad \log\left(\frac{s}{t}\right) = \log(s) - \log(t)$$

$$\log_b\left(\frac{1}{t}\right) = -\log_b(t) \qquad \log_b(s^t) = t\log_b(s).$$

convert between bases: $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)} = \dfrac{\ln(x)}{\ln(b)}$.

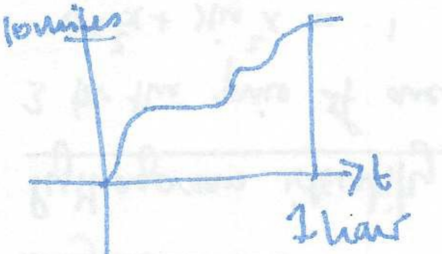
## §2.1 Limits, rates of change, tangent lines




**motivation**: velocity example: driving at constant speed.

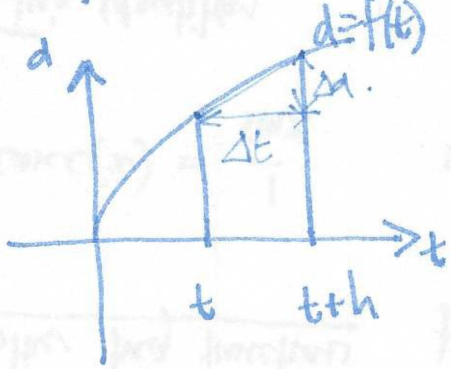
velocity $= \dfrac{\text{distance}}{\text{time}} =$ slope of line

**problem**: what happens if you don't travel at constant speed?



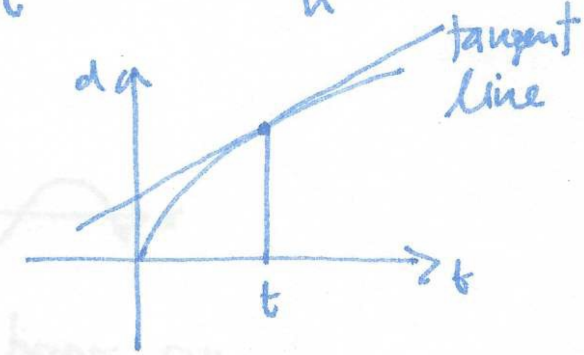

average speed $= \dfrac{\text{distance travelled}}{\text{length of time interval}}$

we can look at average speed over any time interval, including very short ones.




average speed on time interval $[t, t+h]$

is $\dfrac{\Delta d}{\Delta t} = \dfrac{f(t+h)-f(t)}{(t+h)-t} = \dfrac{f(t+h)-f(t)}{h}$




Q: what is the speed at time $t$?
(sometimes called the instantaneous speed)

A: speed is slope of tangent line at $t$.

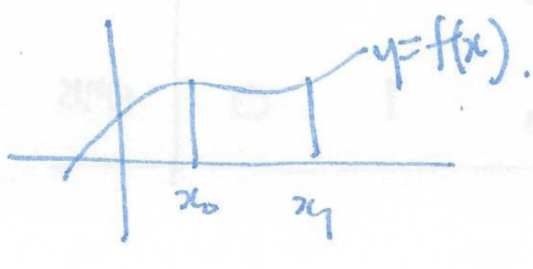
**idea** (hope): as length of interval $[t, t+h]$ gets small, the average speed gets close to the slope of the tangent line.

This works for "nice" functions.

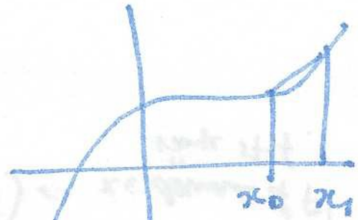
**Observation**: this works for any function $y = f(x)$, not just speed.

**summary** average rate of change over an interval $[x_0, x_1]$

is $\dfrac{f(x_1)-f(x_0)}{x_1 - x_0}$



## § 2.2 Limits




aim: want to find slope of tangent lines

know: average rate of change $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

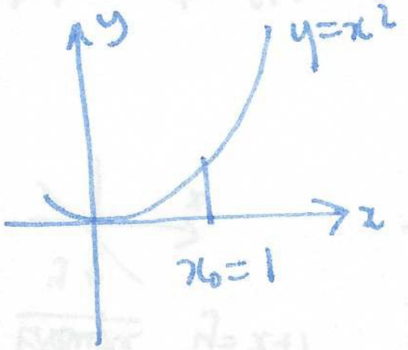
Q: why can't we just set $x_1 = x_0$

A: doesn't work, get $\dfrac{f(x_0)-f(x_0)}{x_0-x_0} \to \dfrac{0}{0}$ undefined.

Observations

① if we draw careful pictures, the average slope gets closer to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculations too:



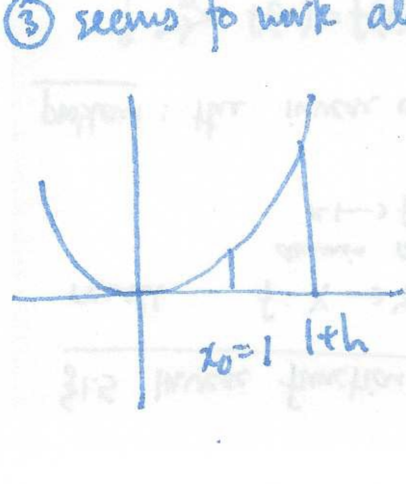

$x_1 = 2: \quad \dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3$

$x_1 = 1.5: \quad \dfrac{f(3/2)-f(1)}{3/2-1} = \dfrac{9/4-1}{1/2} = \dfrac{5}{2} = 2.5$

$x_1 = 1.1 \quad \dfrac{1.21-1}{0.1} = 2.1$

$x_1 = 1.01 \quad \dfrac{1.0201-1}{0.01} = 2.01$

③ seems to work algebraically




average rate of change from $\underset{x_0}{1}$ to $\underset{x_1}{1+h}$

$$= \frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2-1^2}{h} = \frac{1+2h+h^2-1}{h}$$

$$= \frac{2h+h^2}{h} = 2+h$$

$(h \neq 0)$