Math 231 Calculus 1 Fall 14 Final b

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Differentiate the following functions. Do not simplify your answers.

(a) $f(x) = \frac{3}{x} - \frac{2x}{\sqrt{x}} + 5\tan x - 4e^{\pi}$

$f(x) = 3x^{-1} - 2x^{1/2} + 5\tan x - 4e^{\pi}$

$= -3x^{-2} - x^{-1/2} + 5\sec^2 x$

(b) $f(x) = \frac{x^3+1}{x^3-1}$

$f'(x) = \frac{(x^3-1)(3x^2) - (x^3+1)(3x^2)}{(x^3-1)}$

(2) (10 points) Differentiate the following functions. Do not simplify your answers.

(a) $f(x) = \sin(e^x)$

$f'(x) = \cos(e^x) \cdot e^x$

(b) $f(x) = \sqrt{\ln x - 2x^2}$

$f'(x) = \frac{1}{2}(\ln x - 2x^2)^{-1/2} \left(\frac{1}{x} - 4x\right)$

(3) (10 points) Evaluate the following integrals.

(a) $\displaystyle\int \sqrt{x} + 5x^2 + \frac{4}{x} - 3\ dx$

$\frac{2x^{3/2}}{3} + \frac{5}{3}x^3 + 4\ln|x| - 3x + C$

(b) $\displaystyle\int_0^{\pi/6} \cos 3x\ dx$

$\left[\frac{1}{3}\sin 3x\right]_0^{\pi/6} = \frac{1}{3}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = \frac{1}{3}$

(4) (10 points) Evaluate the following integrals.

(a) $\int_0^2 e^{x^4} x^3 \, dx$

$\left[\frac{1}{4} e^{x^4} \right]_0^2 = \frac{1}{4} \left(e^{16} - 1 \right)$

(b) If $\int_0^{10} f(x) \, dx = 15$ and $\int_6^{10} f(x) \, dx = 6$, find $\int_0^6 f(x) \, dx$.

$15 - 6 = 9$.

(5) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\displaystyle\lim_{x\to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$.

l'H: $\displaystyle = \lim_{x\to 2} \frac{2x-5}{2x} = \frac{-1}{4}$

(b) Find $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{x}$.

l'H: $\displaystyle \lim_{x\to 0} \frac{5\cos 5x}{1} = 5$

(c) Find $\displaystyle\lim_{x\to +\infty} \frac{x^2+4}{e^{3x}}$.

l'H: $\displaystyle \lim_{x\to +\infty} \frac{2x}{3e^{3x}}$

l'H: $\displaystyle \lim_{x\to +\infty} \frac{2}{9e^{3x}} = \cancel{\tfrac{2}{9}}\ 0$.

(6) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers. (Hint: one of these questions may require the squeeze theorem.)

(a) Find $\lim_{x\to 3} \frac{1}{2}(x-3)^2 \sin(\frac{1}{x-3})$.

$$-\frac{1}{2}(x-3)^2 \le \frac{1}{2}(x-3)^2 \sin\left(\frac{1}{x-3}\right) \le \frac{1}{2}(x-3)^2$$

$\lim_{x\to 3} \pm\frac{1}{2}(x-3)^2 = 0$ so squeeze thm $\Rightarrow$ $\lim_{x\to 3} \frac{1}{2}(x-3)^2\left(\sin\left(\frac{1}{x-3}\right)\right)^2 = 0$.

(b) Find $\lim_{x\to 1} \frac{e^{2x+1}}{\cos(\pi x)}$. $= \frac{e}{-1} = -e$.

(7) (10 points) Use implicit differentiation to find the tangent line to the curve given by the equation $2y^2 + x^2y + x^3 = 11$ at the point $(1, 2)$.

$4yy' + 2xy + x^2y' + 3x^2 = 0$

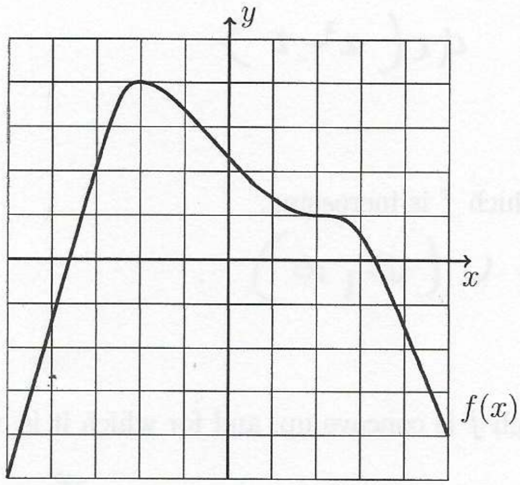
at $(1,2)$: $8y' + 4 + y' + 3 = 0$

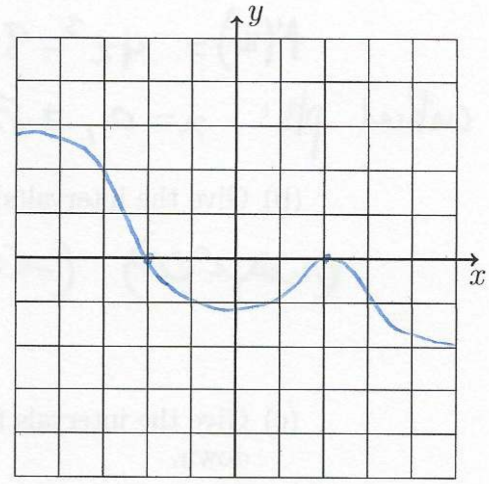
$9y' + 7 = 0$

$y' = -7/9$.

$y - 2 = -\frac{7}{9}(x-1)$.

(8) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) List all the critical points of $f(x)$.
(b) Sketch $y = f'(x)$ on the right hand graph.
(c) Estimate the intervals where $f(x)$ is concave up.

a) $x = -2$, $x = 2$

c) $(0, 2)$.

(9) (10 points) Consider $f(x) = x^4 - 4x^2 + 1$.

(a) Find the derivative of $f(x)$, and find the critical points for $f(x)$.

$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

critical pts: $x = 0, \pm\sqrt{2}$

(b) Give the interval(s) for which f is increasing.

$\cancel{(-\infty, -\sqrt{2})}$ $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$.

(c) Give the intervals for which f is concave up, and for which it is concave down.

$f''(x) = 12x^2 - 8 = 4(3x^2 - 2)$ $x = \pm\sqrt{3/2}$.

concave up $(-\infty, -\sqrt{3/2}) \cup (\sqrt{3/2}, \infty)$ concave down $(-\sqrt{3/2}, +\sqrt{3/2})$.

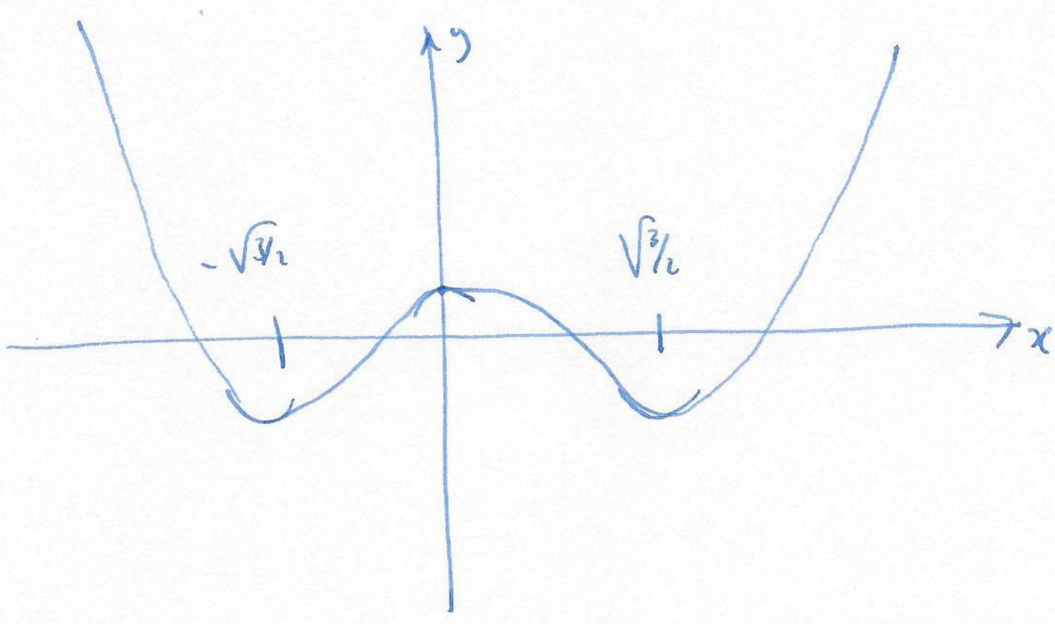
(d) Decide which critical points are maxima, minima, or neither.

$x = -\sqrt{3/2}$ min

$x = 0$ max

$x = +\sqrt{3/2}$ min

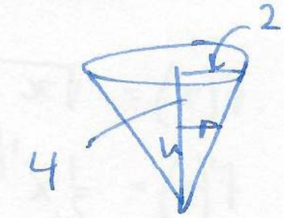
(e) Sketch the graph of $f(x)$.



(10) (10 points) A conical tank has height 4m and radius 2m at the top. Water flows in at a rate of $2.5\text{m}^3/\text{minute}$.

(a) Give a formula for the rate of increase of the water level in terms of its depth h, that is to say for dh/dt. Show the calculation step by step.

(b) How fast is the depth h increasing when the radius reaches 4 meters? Give the numerical value and the unit of measurement.



a) $\left.\begin{array}{l} V=\frac{1}{3}\pi r^2 h \\ h=2r \end{array}\right\}$ $V=\frac{1}{12}\pi h^3$

$$\frac{dV}{dt} = \frac{1}{12}\cdot 3\pi h^2 \frac{dh}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

b) at $h=4$

$$2.5 = \frac{1}{4}\pi\, 16 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2.5\times 4}{16\pi} = \frac{2.5}{4\pi}$$

(11) (10 points)

(a) Use linear approximation to estimate $\sqrt{300}$. Hint: use the fact that $17^2 = 289$.

(b) Compare your answer with value you obtain from your calculator, and find the absolute and percentage errors.

a) $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f(x+h) \approx f(x) + hf'(x)$

$f(300) \approx f(289) + 11 \cdot \frac{1}{2} \frac{1}{\sqrt{289}}$

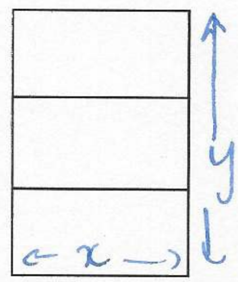
$\sqrt{300} \approx 17 + \frac{11}{34}$

b) $\sqrt{300} \approx 17.32051$

absolute error $= \left| \sqrt{300} - 17\frac{11}{34} \right| = 0.003021336$

percentage error $= 100 \times \frac{0.003021336}{\sqrt{300}} \approx 0.01744369\%$.

(12) (10 points) You wish to build a rectangular window frame in the following shape.



The total length of the frame should be 60ft. The total length is the perimeter plus the two ~~vertical~~ pieces. Determine the width and height which gives the largest area. horizontal

$$A = xy$$
$$60 = 4x + 2y \Leftrightarrow 30 = 2x + y$$

$$A = x(30 - 2x)$$
$$= 30x - 2x^2$$

$$A'(x) = 30 - 4x$$

critical point $x = \frac{30}{4} = \frac{15}{2}$ $\quad y = 15$.