Math 231 Calculus 1 Fall 14 Final a

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Differentiate the following functions. Do not simplify your answers.

(a) $f(x) = \dfrac{2}{x} - \dfrac{3x}{\sqrt{x}} + 4\tan x - 5e^{\pi}$

$$f(x) = 2x^{-1} - 3x^{1/2} + 4\tan x - 5e^{\pi}$$

$$f'(x) = -2x^{-2} - \frac{3}{2}x^{-1/2} + 4\sec^2 x$$

(b) $f(x) = \dfrac{x^3 - 1}{x^3 + 1}$

$$f'(x) = \frac{(x^3+1)(3x^2) - (x^3-1)(3x^2)}{(x^3+1)^2}.$$

$$= \frac{6x^2}{(x^3+1)^2}$$

(2) (10 points) Differentiate the following functions. Do not simplify your answers.

(a) $f(x) = \cos(e^x)$

$f'(x) = -\sin(e^x) . e^x$

(b) $f(x) = \sqrt{x^2 - 2\ln x} = (x^2 - 2\ln x)^{1/2}$

$f'(x) = \frac{1}{2}(x^2 - 2\ln x)^{-1/2} . (2x - \frac{2}{x})$

(3) (10 points) Evaluate the following integrals.

(a) $\displaystyle\int \sqrt{x} + 3x^2 + \frac{5}{x} - 4\ dx$

$= \quad \frac{2x^{3/2}}{3} + x^3 + 5\ln(x) - 4x + C$

(b) $\displaystyle\int_0^{\pi/6} \sin 3x\ dx$

$\left[-\frac{1}{3}\cos 3x\right]_0^{\pi/6} = -\frac{1}{3}\left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right)$

$= -\frac{1}{3}(0-1) = \frac{1}{3}$

(4) (10 points) Evaluate the following integrals.

(a) $\int_0^2 e^{x^4} x^3 \, dx$

$$\left[\frac{1}{4}e^{x^4}\right]_0^2 = \frac{1}{4}\left(e^{16} - 1\right)$$

(b) If $\int_0^8 f(x)\, dx = 20$ and $\int_6^8 f(x)\, dx = 8$, find $\int_0^6 f(x)\, dx$.

$20 - 8 = 12.$

(5) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x\to 3} \dfrac{x^2-5x+6}{x^2-9}$.

l'H : $= \lim_{x\to 3} \dfrac{2x-5}{2x} = \dfrac{1}{6}$

(b) Find $\lim_{x\to 0} \dfrac{\sin 3x}{x}$.

l'H : $= \lim_{x\to 0} \dfrac{\cos 3x \cdot 3}{1} = 3$

(c) Find $\lim_{x\to +\infty} \dfrac{x^2+5}{e^{2x}}$.

l'H : $= \lim_{x\to +\infty} \dfrac{2x}{e^{2x}\cdot 2}$

l'H : $= \lim_{x\to +\infty} \dfrac{2}{4e^{2x}} = 0.$

(6) (10 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers. (Hint: one of these questions may require the squeeze theorem.)

(a) Find $\lim_{x\to 5} \frac{1}{2}(x-5)^2 \sin(\frac{1}{x-5})$.

Squeeze Thm :

$$-\frac{1}{2}(x-5)^2 \leq \frac{1}{2}(x-5)^2 \sin\left(\frac{1}{x-5}\right) \leq \frac{1}{2}(x-5)^2$$

$\lim_{x\to 5} \pm\frac{1}{2}(x-5)^2 = 0 \Rightarrow \lim_{x\to 5} \frac{1}{2}(x-5)^2 \sin\left(\frac{1}{x-5}\right) = 0.$

(b) Find $\lim_{x\to 1} \frac{e^{2x-1}}{\cos(\pi x)}$. $= \frac{e^1}{-1} = -e$

(7) (10 points) Use implicit differentiation to find the tangent line to the curve given by the equation $y^2 + x^2y + 2x^3 = 8$ at the point $(1, 2)$.

$2yy' + 2xy + x^2y' + 6x^2 = 0$

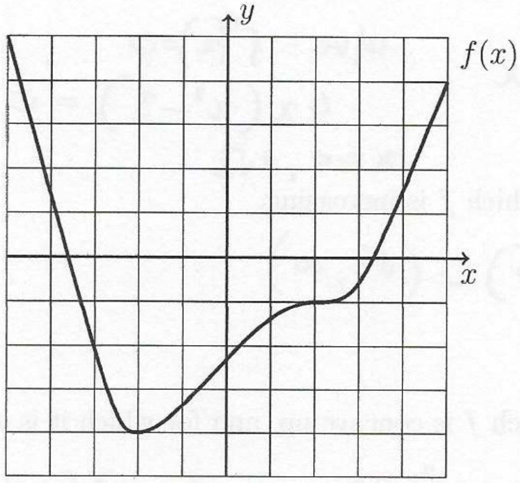
at $(1,2)$: $4y' + 4 + y' + 6 = 0$

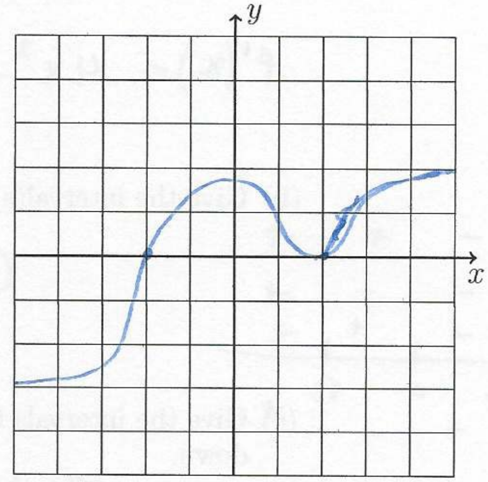
$5y' + 10 = 0$

$y' = -2$

$y - 2 = -2(x-1)$

(8) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) List all the critical points of $f(x)$.
(b) Sketch $y = f'(x)$ on the right hand graph.
(c) Estimate the intervals where $f(x)$ is concave up.

a) $x = -2 \quad x = 2$

c) $(-5, 0) \cup (3, 5)$

(9) (10 points) Consider $f(x) = x^4 - 6x^2 - 1$.

(a) Find the derivative of $f(x)$, and find the critical points for $f(x)$.

$f'(x) = 4x^3 - 12x$

solve: $f'(x) = 0$

$4x(x^2 - 3) = 0$

$x = 0, \pm\sqrt{3}$.

(b) Give the interval(s) for which f is increasing.

$4x$	$-$	$-$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
	$-\sqrt{3}$	0	$\sqrt{3}$	
$f'(x)$	$-$	$+$	$-$	$+$

$(\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

(c) Give the intervals for which f is concave up, and for which it is concave down.

$f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$.

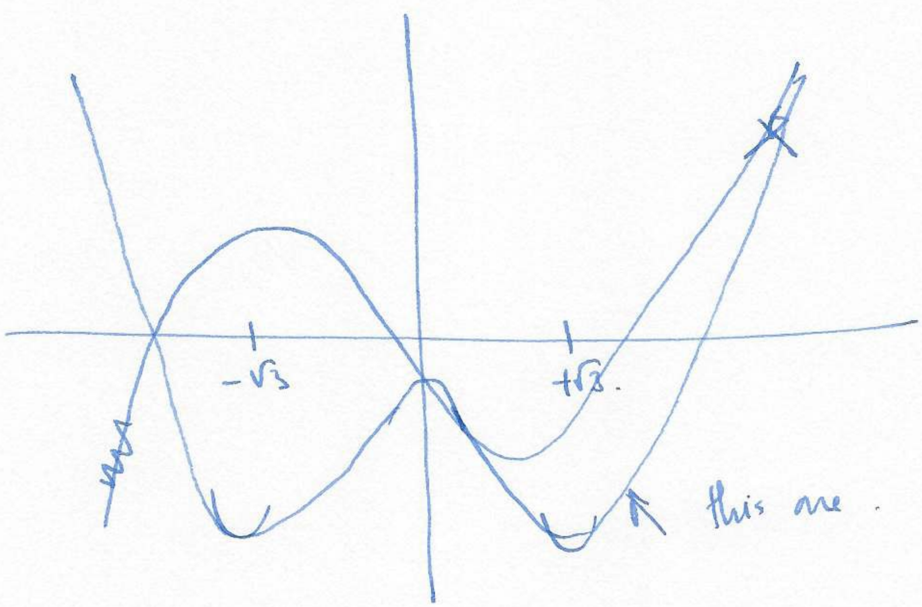
concave up $(-\infty, -1) \cup (1, \infty)$ concave down $(-1, 1)$.

(d) Decide which critical points are maxima, minima, or neither.

0 max

$\pm\sqrt{3}$ min

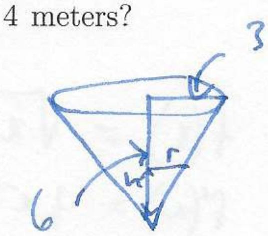
(e) Sketch the graph of $f(x)$.



(10) (10 points) A conical tank has height 6m and radius 3m at the top. Water flows in at a rate of $3.5\text{m}^3/\text{minute}$.

(a) Give a formula for the rate of increase of the water level in terms of its depth h, that is to say for dh/dt. Show the calculation step by step.

(b) How fast is the depth h increasing when the radius reaches 4 meters? Give the numerical value and the unit of measurement.



a) $V = \frac{1}{3}\pi r^2 h$ $\quad h = 2r \leftrightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi \frac{1}{4} h^3$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ $\qquad \frac{dh}{dt} = \frac{3.5 \times 4}{\pi h^2}$

b) at $h = 4$ $\quad \frac{dh}{dt} = \frac{3.5 \times 4}{16\,\cancel{64}\pi}$

(11) (10 points)

(a) Use linear approximation to estimate $\sqrt{200}$. Hint: use the fact that $14^2 = 196$.

(b) Compare your answer with value you obtain from your calculator, and find the absolute and percentage errors.

a) $f(x) = \sqrt{x}$ $\qquad f(x+h) \approx f(x) + f'(x)h$

$f'(x) = \frac{1}{2}x^{-1/2}$

$\sqrt{200} \approx \sqrt{196} + \frac{1}{2\sqrt{196}} \cdot (4)$

$14 + \frac{4}{2 \times 14} = 14 + \frac{1}{28} = 14\frac{1}{28}$

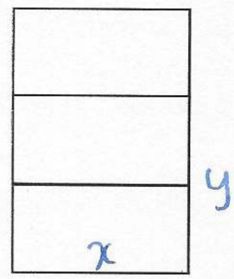
b)

$\sqrt{200} \simeq 14.14214.$

absolute error $\left|\left(\sqrt{200} - 14\frac{1}{28}\right)\right| = 0.1064213.$

percentage error $\frac{0.1064213}{\sqrt{200}} \times 100 = 0.7525125\%.$

(12) (10 points) You wish to build a rectangular window frame in the following shape.



The total length of the frame should be 40ft. The total length is the perimeter plus the two ~~vertical~~ pieces. Determine the width and height which gives the largest area. horizontal

$4x+2y=40 \Leftrightarrow 2x+y=20.$

$A = xy = x(20-2x) = 20x-2x^2$

$\frac{dA}{dx} = 20-4x$ critical point solve $A'(x)=0$:

$20-4x=0 \Rightarrow x=5.$

$y=10.$